97. A Remark on Quasi-Invariant Measure

By Shozo KOSHI and Yasuji TAKAHASHI
Hokkaido University

(Comm. by Kôsaku YOSIDA, M. J. A., Sept. 12, 1974)

In this short note, we shall give a simple proof of Dao-Xing's theorem concerning quasi-invariant measures [1]. Although essential part of this proof is nearly equal to the original one, the assumptions of the original theorem are slightly weakened by this proof. Let $E\subset F$ be linear topological spaces and the linear topological space E be second category with countable basis of nbds (neighbourhoods) of 0, and let the injection map $E\to F$ be continuous.

Theorem. *Let $\mathfrak{B}$ be a σ-algebra of F which is invariant under E and contains all cylinder sets induced by F^* (dual of F). If μ is a non-trivial E-quasi-invariant measure on $(F,\mathfrak{B})$, then there exist a nbd V of 0 in E and a positive real number C such that*

$$\sup_{h\in V}|f(h)|\leqq C\int|f(x)|\,d\mu(x)\qquad \text{for all } f\in F^*.$$

Proof. Assume the contrary. Then, for all positive integer n, we can find an $f_n\in F^*$ and a nbd V_n of 0 in E such that

$$\sup_{h\in V_n}|f_n(h)|>n\int|f_n(x)|\,d\mu(x),$$
$$V_1\supset V_2\supset\cdots\supset V_n\supset V_{n+1}\supset\cdots,$$

and $\{V_n\}$ is a basis of nbds of 0 in E.

Clearly we have $\int|f_n(x)|\,d\mu(x)<\infty$. We shall prove that $0<\int|f_n(x)|\,d\mu(x)$. If not, $f_n(x)=0$ almost everywhere on F. For $A_n=\{x\in F\,;\,f_n(x)=0\}$, we have $\mu(A_n^c)=0$.

Since μ is E-quasi-invariant, we have $\mu(A_n^c+h)=0$ for all $h\in E$, i.e. $\mu([A_n\cap(A_n+h)]^c)=0$.

Since μ is non-trivial, we have $\mu(A_n\cap(A_n+h))>0$. Hence, there exists $x\in F$ with $x\in A_n$ and $x-h\in A_n$. Thus, we have

$$f_n(h)=f_n(x)-f_n(x-h)=0\qquad\text{for all } h\in E.$$

This is a contradiction.

Let $a_n=\int|f_n(x)|\,d\mu(x)$ and consider $l^1(a_n)=\left\{\xi=(\xi_n)\,;\,\sum_n|\xi_n|\times\int|f_n(x)|\,d\mu(x)<\infty\right\}$. For $\xi=(\xi_n)\in l^1(a_n)$, we find

$$q_\xi(h)=\sum_n|\xi_nf_n(h)|<\infty\qquad\text{for all } h\in E$$

by the same argument which is already shown before.

Since $q_\xi(h)$ is sub-additive and lower semi-continuous on E and E is second category, there exists a nbd V of 0 in E with

$$\sup_{h \in V} q_\xi(h) < \infty.$$

Since $\{V_n\}$ is a basis of nbds of 0, there exists $V_m \subset V$.

Let $p_n(\xi) = \sup_{h \in V_n} q_\xi(h)$ for $\xi \in l^1(a_n)$. Then, $0 \leqq p_n(\xi) \leqq \infty$ for all $\xi \in l^1(a_n)$ and for $\xi \in l^1(a_n)$ there exists m with $p_m(\xi) < \infty$.

Using Gelfand's theorem (since $l^1(a_n)$ is a Banach space), we see that

$$p_m(\xi) \leqq C \|\xi\|_1 \qquad \text{for some } m \text{ and } C > 0.$$

Hence, we have

$$\sup_{h \in V_m} |f_n(h)| \leqq C \int |f_n(x)|\, d\mu(x) \qquad \text{for all } n.$$

This is a contradiction. q.e.d.

Remark. As is shown in this proof, we can take off the regularity and local finiteness of μ from the assumptions of Dao-Xing's original theorem.

Reference

[1] Xia Dao-Xing: Measure and Integration Theory on Infinite-Dimensional Spaces. Academic Press, New York (1972).