184. *A Remark on Picard Principle*

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A nonnegative locally Hölder continuous function $P(z)$ on $0<|z|\leq 1$ will be referred to as a *density* on the punctured unit disk $\Omega: 0<|z|<1$ with a singularity at $z=0$, removable or genuine. The *elliptic dimension* of a density P on Ω at $z=0$, $\dim P$ in notation, is the dimension of the half module of nonnegative solutions u of the equation $\Delta u=Pu$ on Ω with vanishing boundary values on $|z|=1$. After Bouligand we say that the *Picard principle* is valid for a density P at $z=0$ if $\dim P=1$. That the Picard principle is valid for the density $P(z)\equiv 0$, i.e. for the harmonic case, is the well known classical result. Less trivial examples are $P(z)=|z|^{-\lambda}$ ($\lambda\in(-\infty,2]$) (cf. [2]) and densities $P(z)$ with the property

$$\int_{\Omega-E} P(z)\log\frac{1}{|z|}dxdy<\infty \qquad (z=x+iy)$$

where $E=E_P$ is a closed subset of Ω thin at $z=0$ (cf. [3]). These examples suggest that singularities of densities $P(z)$ at $z=0$ for which the Picard principle is valid are 'not so wild'. In view of this one might be tempted to say that if the Picard principle is valid for two densities P_j $(j=1,2)$, then it is also valid for the density P_1+P_2. The purpose of this note is to stress the complexity of the Picard principle by showing that the above intuition is wrong. Namely we shall prove the following

Theorem. *There exists a pair of densities P_j $(j=1,2)$ on Ω such that the Picard principle is valid for P_j $(j=1,2)$ at $z=0$ but invalid for the density P_1+P_2 at $z=0$.*

Actually densities P_j $(j=1,2)$ we are going to construct as stated in the above theorem are *rotation free* in the sense that $P_j(z)=P_j(|z|)$ on Ω, and satisfy $\dim P_j=1$ $(j=1,2)$ and $\dim(P_1+P_2)=\mathfrak{c}$ (the cardinal number of continuum). This also shows the invalidity of subadditivity of elliptic dimensions, i.e. the following inequality does not hold in general:

$$\dim(P_1+P_2)\leq\dim P_1+\dim P_2.$$

1. To construct the required P_j we need to consider auxiliary functions $s(t;\lambda,\mu)$ and $c(t;\lambda,\mu)$ which are modifications of trigonometric functions. Let $\lambda\in[1,2)$ and $\mu\in\boldsymbol{R}$ (the real number field). Consider mutually disjoint closed intervals

$$\sigma_n(\lambda,\mu)=[(2n-(\lambda+1)/2+\mu)\pi,(2n+(\lambda-1)/2+\mu)\pi]$$

for $n\in N$ (the set of integers) and the open set

$$\tau(\lambda,\mu)=R-\bigcup_{n\in N}\sigma_n(\lambda,\mu).$$

We define functions $s(t;\lambda,\mu)$ and $c(t;\lambda,\mu)$ on R by

$$s(t;\lambda,\mu)=\begin{cases}\sin(\lambda^{-1}(t-(2n-(\lambda+1)/2+\mu)\pi)) & (t\in\sigma_n(\lambda,\mu),n\in N);\\ 0 & (t\in\tau(\lambda,\mu))\end{cases}$$

and similarly

$$c(t;\lambda,\mu)=\begin{cases}\cos(\lambda^{-1}(t-(2n-(\lambda+1)/2+\mu)\pi)) & (t\in\sigma_n(\lambda,\mu),n\in N);\\ 0 & (t\in\tau(\lambda,\mu)).\end{cases}$$

The function $s(t;\lambda,\mu)$ is continuous but $c(t;\lambda,\mu)$ is not. However $c^2(t;\lambda,\mu)$ is continuous and

$$s^2(t;\lambda,\mu)+c^2(t;\lambda,\mu)=1$$

on R. The function $s(t;\lambda,\mu)$ itself is not of class C^1 but $s^2(t;\lambda,\mu)$ is, and

$$\frac{d}{dt}s^2(t;\lambda,\mu)=2\lambda^{-1}s(t;\lambda,\mu)c(t;\lambda,\mu)$$

on R which is equal to $\lambda^{-1}\sin(2\lambda^{-1}(t-(2n-(\lambda+1)/2+\lambda)\mu)\pi))$ on $\sigma_n(\lambda,\mu)$ $(n\in N)$ and 0 on $\tau(\lambda,\mu)$. Thus $ds^2(t;\lambda,\mu)/dt$ is less than or equal to 1 in the absolute value. We set

$$s_1(t)=s(t;1,1),\qquad c_1(t)=c(t;1,1)$$

and similarly

$$s_2(t)=s(t;3/2,0),\qquad c_2(t)=c(t;3/2,0).$$

2. With the aid of auxiliary functions s_j and c_j we define

$$\begin{aligned}P_j(z)=|z|^{-2}\Big\{&\Big(\log\frac{1}{|z|}\Big)^4 s_j^4\Big(\log\frac{1}{|z|}\Big)+2\Big(\log\frac{1}{|z|}\Big)c_j^2\Big(\log\frac{1}{|z|}\Big)\\ &+2\Big(1+\log\frac{1}{|z|}\Big)\Big(\log\frac{1}{|z|}\Big)^2 s_j^2\Big(\log\frac{1}{|z|}\Big)\\ &+\Big(\log\frac{1}{|z|}\Big)^2\Big(1-3^{-1}s_j\Big(\log\frac{1}{|z|}\Big)c_j\Big(\log\frac{1}{|z|}\Big)\Big)\Big\}\end{aligned}$$

for $j=1,2$. These are certainly rotation free densities on Ω. We shall prove that $\dim P_j=1$ $(j=1,2)$ and $\dim(P_1+P_2)=\mathfrak{c}$.

3. Before proceeding to the proof of the assertion in no. 2 we need to make some preparations. Let $P(z)$ be a *rotation free density* on Ω, i.e. $P(z)=P(|z|)$. Then $\dim P=1$ or $\mathfrak{c}$ (cf. [2]). The *associated function* $Q(t)$ to $P(z)$ is the function on $[0,\infty)$ defined by

$$Q(t)=e^{-2t}P(e^{-t}).$$

The *Riccati component* a_Q of Q is the unique nonnegative solution of the equation

$$-\frac{d}{dt}a(t)+a^2(t)=Q(t)$$

on $[0,\infty)$ (cf. [4]). It is known (cf. [1]) that $\dim P=1$ ($\mathfrak{c}$, resp.) is characterized in terms of a_Q by

$$\int_0^\infty (a_Q(t)+1)^{-1}dt=\infty \qquad (<\infty, \text{ resp.}).$$

4. We are ready to prove the assertion in no. 2. Let Q_j be the associated function to P_j ($j=1,2$). Then Q_1+Q_2 is the associated function to P_1+P_2. In view of no. 3 all we have to show is that

$$\int_0^\infty (a_{Q_j}(t)+1)^{-1}dt=\infty \qquad (j=1,2), \tag{1}$$

which is equivalent to $\dim P_j=1$ ($j=1,2$), and

$$\int_0^\infty (a_{Q_1+Q_2}(t)+1)^{-1}dt<\infty, \tag{2}$$

which is equivalent to $\dim(P_1+P_2)=\mathfrak{c}$. On using relations in no. 1 concerning s_j and c_j it is easily checked that

$$a_{Q_j}(t)=t^2s_j^2(t)+t+1 \qquad (j=1,2).$$

Therefore the integrand in (1) is $t+1$ on the disjoint countable union of open intervals with the constant positive length and we conclude that (1) is true.

We recall that $Q\to a_Q$ is an order preserving operator (cf. [4]). Set $Q\equiv Q_1+Q_2+2a_{Q_1}\cdot a_{Q_2}$ which is the associated function to a rotation free density $P(z)\equiv|z|^{-2}Q(-\log|z|)$ on Ω. It is easily seen that $a_{Q_1}+a_{Q_2}=a_Q$ and hence $Q\geq Q_1+Q_2$ implies $a_{Q_1}+a_{Q_2}=a_Q\geq a_{Q_1+Q_2}$. Similarly $Q_1+Q_2\geq Q_j$ ($j=1,2$) implies that $a_{Q_1+Q_2}\geq a_{Q_j}$ ($j=1,2$). Therefore

$$2a_{Q_1+Q_2}\geq a_{Q_1}+a_{Q_2}\geq a_{Q_1+Q_2}$$

and a fortiori (2) is equivalent to

$$\int_0^\infty (a_{Q_1}(t)+a_{Q_2}(t)+1)^{-1}dt<\infty. \tag{3}$$

Here $a_{Q_1}(t)+a_{Q_2}(t)=t^2(s_1^2(t)+s_2^2(t))+2t+2$. Observe that $s_1^2+s_2^2$ has a positive infimum on $\boldsymbol{R}$ and hence on $[0,\infty)$. Therefore the integrand in (3) is dominated by a positive constant multiple of $(t+1)^{-2}$ and we conclude that (3) is valid.

References

[1] M. Kawamura and M. Nakai: A test of Picard principle for rotation free densities. II (to appear).
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[3] ——: A test for Picard principle (to appear in Nagoya Math. J., **56**).
[4] ——: A test of Picard principle for rotation free densities (to appear).