25. On Decompositions of Linear Mappings among Operator Algebras

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1. Introduction. Let φ be a $B(H)$-valued function on a set X where $B(H)$ is the algebra of all (bounded linear) operators on a Hilbert space H, and (S) be a property on such φ's. A (closed) subspace M of H *(S)-reduces* φ if M reduces $\varphi(x)$ for all $x \in X$ and $\varphi(x)|M \in (S)$ where $\psi \in (S)$ if ψ has (S). For a subspace N reducing all $\varphi(x)$, the function $\varphi(x)|N$ is *completely non-(S)* if there is no non-zero subspace which (S)-reduces the function.

A strongly closed set P of projections of a von Neumann algebra A is a *Szymanski family* if P satisfies the following conditions (cf. [6]):

(1) *If $e, f \in P$ then $e \wedge f \in P$,*

(2) *If $e, f \in P$ and $ef=0$ then $e+f \in P$,*

(3) *If $e, f \in P$ and $e \geq f$ then $e-f \in P$*

and

(4) *If $e \in P$, $f \in \mathrm{proj}\,(A)$ and $e \sim f$ (mod. A) then $f \in P$.* P is called *hereditary* if it satisfies

(5) *If $e \in P$, $f \in \mathrm{proj}\,(A)$ and $e \geq f$ then $f \in P$.*

If P is a hereditary Szymanski family, then P is a principal ideal of the lattice $L=\mathrm{proj}\,(A)$, cf. [9, Lemma 2], and the largest element e_0 of P is central according to [9, Theorem 5]. Recently Y. Kato and S. Maeda [8] proved that the localization of e_0 in the center of L has a purely lattice theoretic character. Summing up:

Theorem 1. *If P is a Szymanski family in a von Neumann algebra A, then there exists the largest projection e_0 of P in the center of A.*

Let $A=(\varphi(X) \cup \varphi(X)^*)'$ where B' is the commutant of B. A property (S) is called a *Szymanski property* if

$$P=\{e \in \mathrm{proj}\,(A) : \varphi(\cdot)\,|\,eH \in (S)\}$$

is a hereditary Szymanski family. Szymanski [9] proved the following general decomposition theorem for operator valued functions.

Theorem 2. *If (S) is a Szymanski property, then there exists the largest (S)-reducing subspace e_0H such that $\varphi(\cdot)e_0 \in (S)$, and $\varphi(\cdot)e_0^{\perp}$ is completely non-(S).*

In the present note we shall show that these theorems are applicable to operator algebras. We shall treat the decomposition of expectations, operator valued measures, automorphisms and linear mappings in §§ 2–5. In § 6 we shall apply Theorem 1 to show that every von Neumann algebra A with a subalgebra B splits into the direct sum of the part continuous over B and a part discrete over B when B is contained in the center of A, which is discussed by M. Choda [1], [2] and [3].

2. Expectations. Let A be a von Neumann algebra and B a von Neumann subalgebra of A. An expectation ε of A onto B is called *normal* if it satisfies $\varepsilon(\sup x_\alpha) = \sup \varepsilon(x_\alpha)$ for every uniformly bounded increasing net $\{x_\alpha\}$ of positive elements in A. ε is *singular* if it is completely non-normal. An expectation ε is called *abelian* if it satisfies $\varepsilon(xy) = \varepsilon(yx)$ for each $x, y \in A$. J. Tomiyama [10] obtained the following decomposition of expectations:

Theorem 3. *For an expectation ε of A onto B, there exists the largest projection e_0 in the center of B such that $\varepsilon(e_0 \cdot)$ is normal and $\varepsilon(e_0^\perp \cdot)$ is singular.*

Proof. If is sufficient to prove that normality is a Szymanski property. Let $\{x_\alpha\}$ be a uniformly bounded increasing net of positive elements in A. For $f \in \mathrm{proj}\,(B')$ which is equivalent to $e \in P$ (mod. B'), there is $v \in B'$ such that:

$$v^*v = e,\ vv^* = f \quad \text{and} \quad \sup \varepsilon(x_\alpha)v^*v = \varepsilon(\sup x_\alpha)v^*v.$$

And we have $\sup \varepsilon(x_\alpha)f = v \sup \varepsilon(x_\alpha)v^* = v\varepsilon(\sup x_\alpha)v^* = \varepsilon(\sup x_\alpha)f$, so $f \in P$. Let $\{e_\beta\}$ is a net in P converging strongly to $e \in \mathrm{proj}\,(B')$. Then we have $\sup \varepsilon(x_\alpha)e = \text{s-lim} \sup \varepsilon(x_\alpha)e_\beta = \text{s-lim}\ \varepsilon(\sup x_\alpha)e_\beta = \varepsilon(\sup x_\alpha)e$, which shows that $e \in P$.

Similarly we obtain the abelian part of an expectation, since abelianness is a Szymanski property.

Proposition 4. *For an expectation ε of A onto B, there exists the largest projection e_0 in the center of B such that $\varepsilon(e_0 \cdot)$ is abelian and $\varepsilon(e_0^\perp \cdot)$ is completely non-abelian.*

3. Operator valued measures. Let (X, B) be a measurable space. An operator valued function a on B is called an *operator valued measure* if $(a(\cdot)h, k)$ is a measure on B for every $h, k \in H$. An operator valued measure is a *semi-spectral* (resp. *spectral*) *measure* if a is positive operator (resp. projection) valued. These properties are Szymanski properties. Hence we have by Theorem 2:

Proposition 5. *For an operator valued measure (resp. semi-spectral measure) on B, there exists the largest projection e_0 in the center of $A = (a(B) \cup a(B)^*)'$ such that $a(\cdot)e_0$ is a semi-spectral measure (resp. spectral measure) and $a(\cdot)e_0^\perp$ is completely non-semi-spectral (resp. completely non-spectral)*

Let (X, B, μ) be a measure space. An operator valued measure a

on B is called *absolutely continuous with respect to* μ if $(a(\cdot)h, k)$ is absolutely continuous with respect to μ for every $h, k \in H$. If a is completely non-absolutely continuous w. r. t. μ, then a is called *singular* w. r. t. μ. Clearly the absolute continuity is a Szymanski property. Hence we can conclude that there exists the largest projection e_0 in the center of $A=(a(B)\cup a(B)^*)'$ such that $a(\cdot)e_0$ is absolutely continuous and $a(\cdot)e_0^\perp$ is singular. Especially for a contraction operator b, there exists a semi-spectral measure a on the Borel family B on the unit circle such that

$$b=\int xda(x).$$

By the above discussion, we have a decomposition of contractions, cf. [5: p. 56, ex. 7]: If b is a contraction, then there is a unique reducing subspace such that $b\,|\,M$ is absolutely continuous (i.e., its semi-spectral measure is absolutely continuous) and $b\,|\,M^\perp$ is singular.

4. **Automorphisms.** Let α be an (*-) automorphism on a von Neumann algebra with the center $Z=A\cap A'$, and the *fixed algebra* B of α:

$$B=\{x\in A: \alpha(x)=x\}.$$

If for a property (S) on an automorphism α, the set

$$P=\{e\in \text{proj}\,(Z\cap B): \alpha(\cdot)e\in (S)\}$$

is a hereditary Szymanski family, then we can decompose α into (S)-part and completely non-(S)-part. The following one is such a property:

(6) α is *freely acting*, i.e., *if* $ax=\alpha(x)a$ *for every* $x\in A$ *then* $a=0$.

It is known that the free action is the complementary concept of innerness. So we have the following theorem due to R. R. Kallman [7].

Theorem 6. *For an automorphism α on A, there exists the largest projection e_0 in $Z\cap B$ such that $\alpha(e_0\cdot)$ is freely acting and $\alpha(e_0^\perp\cdot)$ is inner.*

5. **σ-weakly continuous linear mappings.** Let A and B be von Neumann algebras, and φ be a linear mapping from A into B. If for a property (S) on such φ's, the set

$$P=\{e\in \text{proj}\,(A\cap A'): \varphi(e\cdot)\in (S)\}$$

is a hereditary Szymanski family, then there is the largest projection e_0 in P.

Theorem 7. *For a σ-weakly continuous *-preserving linear mapping φ on A into B, there exists the largest projection e_0 in the center of A such that $\varphi(e_0\cdot)$ is positive and $\varphi(e_0^\perp\cdot)$ is negative.*

Proof. P is strongly closed since φ is σ-weakly continuous. For every $x\in A$, $e\in P$ and $f\in \text{proj}\,(A\cap A')$ such that $e\geq f$, we have

$$\varphi(fx^*x)=\varphi(efx^*x)=\varphi(e(xf)^*(xf))\geq 0,$$

and $f \in P$. Hence P is a hereditary Szymanski family.

Theorem 7 is reduced to [4: Chap. I. §4. ex. 10] for linear functionals.

6. Types over von Neumann subalgebras. Let B be a von Neumann subalgebra of a von Neumann algebra A, B^c be the *relative commutant* $B' \cap A$ of B in A, and $\bar{e}$ the *B-support* of $e \in \operatorname{proj}(A)$, that is,

$$\bar{e} = \inf\{f \in \operatorname{proj}(B) : f \geq e\}.$$

A projection $e \in A$ is *abelian over B* if $e \in B^c$ and $Ae = Be$. A von Neumann algebra A is *continuous over B* if A contains no non zero projections abelian over B. Also A is *discrete over B* if there is $e \in \operatorname{proj}(A)$ which is abelian over B and $\bar{e} = 1$. Among others, M. Choda [3] proved the following:

Theorem 8. *If B is contained in the center of A, then there exists the largest projection $e_0 \in B \cap B'$ such that Ae_0 is continuous over Be_0 and $Ae_0^{\perp}$ is discrete over $Be_0^{\perp}$.*

Proof. It is sufficient to prove that the set

$$P = \{e \in \operatorname{proj}(B \cap B') : Ae \text{ is continuous over } Be\}$$

is a hereditary Szymanski family, since continuity and discreteness over B is the complementary properties. If $\{e_\beta\}$ is a net in P which converges strongly to $e \in B \cap B'$, and if Ae is not continuous over Be, then there is non-zero projection f abelian over Be, and e_β such that $e_\beta f \neq 0$, since $e_\beta f$ converges to $ef = f \neq 0$ strongly. By $(Be_\beta)^c \ni e_\beta f \leq f$, we have $e_\beta f$ is abelian over Be and hence over Be_β, which is a contradiction. Therefore P is a hereditary Szymanski family.

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