22. On an Asymptotic Property of Spectra of a Random Difference Operator

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We consider a discrete version of the Schrödinger operator with a random potential q:

(1) $$(H^{\omega}u)(a) = -(H^0u)(a) + q(a,\omega)u(a), \qquad a \in Z^{\nu}.$$

Here u is a function on the ν-dimensional lattice space Z^{ν}, H^0 is a second order difference operator defined by

(2) $$(H^0u)(a) = \frac{\sigma^2}{2}\sum_{i=1}^{\nu}\{u(a_1,\cdots,a_i-1,\cdots,a_{\nu}) - 2u(a) + u(a_1,\cdots,a_i+1,\cdots,a_{\nu})\}, \qquad a \in Z^{\nu},$$

with a positive constant σ and $\{q(a,\omega); a \in Z^{\nu}\}$ is a family of random variables defined on a certain probability space $(\Omega, \mathcal{B}, P)$. The only assumption we make on random variables $\{q(a,\omega); a \in Z^{\nu}\}$ is that they form a stationary random field. Their common distribution function will be denoted by $F(x)$:

(3) $$F(x) = P(q(0) \leqq x), \qquad x \in R^1.$$

Denote by $L^2(Z^{\nu})$ the space of all square summable functions on Z^{ν} with inner product $(u,v) = \sum_{a\in Z^{\nu}} u(a)v(a)$. Since H^0 is a bounded symmetric operator on $L^2(Z^{\nu})$, it is easy to see that the operator H^{ω} restricted to the space $C_0(Z^{\nu})$ of functions with finite supports is essentially self-adjoint and that its self-adjoint extension A^{ω} can be described as follows:

(4) $$\begin{cases} \mathcal{D}(A^{\omega}) = \{u \in L^2(Z^{\nu}); H^{\omega}u \in L^2(Z^{\nu})\} \\ A^{\omega}u = H^{\omega}u \qquad u \in \mathcal{D}(A^{\omega}). \end{cases}$$

Let $\{E^{\omega}_{\lambda}; \lambda \in R^1\}$ be the resolution of the identity associated with A^{ω}: $A^{\omega} = \int_{-\infty}^{\infty} \lambda dE^{\omega}_{\lambda}$.

It turns out that $(E^{\omega}_{\lambda}I_0, I_0)$ is measurable in $\omega \in \Omega$ where $I_0(a) = \delta_0(a)$ $a \in Z^{\nu}$. So we can define

(5) $$\rho(\lambda) = \langle (E^{\omega}_{\lambda}I_0, I_0)\rangle$$

where $\langle\ \ \rangle$ is the expectation with respect to the probability measure P. ρ is called the spectral distribution function of the ensemble of operators $\{H^{\omega}; \omega \in \Omega\}$. Our present aim is to show that ρ and the distribution function F of $q(0)$ have the same tails asymptotically in the following sense.

Theorem. *The next two conditions are equivalent to each other for $\alpha>1$ and $A>0$.*

$$\lim_{x\to-\infty}\frac{\log F(x)}{|x|^{\alpha}}=-A. \tag{6}$$

$$\lim_{\lambda\to-\infty}\frac{\log \rho(\lambda)}{|\lambda|^{\alpha}}=-A. \tag{7}$$

We can prove this theorem quite easily if we use the following two lemmas. We introduce a time continuous Markov process $\dot{M}=(\dot{\Omega},\dot{\mathcal{B}},\dot{X}_t,\dot{P}_a)_{a\in Z^{\nu}}$ on Z^{ν} whose generator is H^0.

Lemma 1.

$$t\int_{-\infty}^{\infty}e^{-tx}\rho(x)dx=\dot{E}_0\Big(\Big\langle\exp\Big(-\int_0^t q(\dot{X}_s)ds\Big)\Big\rangle;\ \dot{X}_t=0\Big)(\leqq\infty),\qquad t>0, \tag{8}$$

$\dot{E}_0$ being the expectation with respect to $\dot{P}_0$.

Lemma 2. *Let $H(\lambda)$ be a non-decreasing function such that $H(-\infty)=0$ and $G(t)$ be its Laplace transform:*

$$G(t)=\int_{-\infty}^{\infty}e^{-t\lambda}dH(\lambda). \tag{9}$$

Then the following two statements are equivalent:

$$\text{ilm}_{\lambda\to-\infty}\frac{\log H(\lambda)}{|\lambda|^{\alpha}}=-A\qquad \alpha>1,\ A>0, \tag{10}$$

$$\lim_{t\to\infty}\frac{\log G(t)}{t^{\gamma}}=B\qquad \gamma>1,\ B>0. \tag{11}$$

Here α,γ,A and B are related by

$$\gamma=\frac{\alpha}{\alpha-1}\Big(\alpha=\frac{\gamma}{\gamma-1}\Big)\ B=(\alpha-1)\alpha^{\alpha/(1-\alpha)}A^{1/(1-\alpha)}(A=(\gamma-1)\gamma^{\gamma/(1-\gamma)}B^{1/(1-\gamma)}).$$

Proof of Theorem. It holds that

$$\begin{aligned}&e^{-\nu\sigma^2 t}\int_{-\infty}^{\infty}e^{-t\lambda}dF(\lambda)\leqq\dot{E}_0\Big(\Big\langle\exp\Big(-\int_0^t q(\dot{X}_s)ds\Big)\Big\rangle;\ \dot{X}_t=0\Big)\\&\leqq\int_{-\infty}^{\infty}e^{-t\lambda}dF(\lambda),\end{aligned} \tag{12}$$

because we have

$$\begin{aligned}&\dot{E}_0\Big(\Big\langle\exp\Big(-\int_0^t q(\dot{X}_s)ds\Big)\Big\rangle;\ \dot{X}_t=0\Big)\\&\qquad\geqq\dot{E}_0(\langle\exp(-tq(0))\rangle;\ \dot{X}_s=0\ 0\leqq s\leqq t)=e^{-\nu\sigma^2 t}\int_{-\infty}^{\infty}e^{-t\lambda}dF(\lambda)\end{aligned}$$

and furthermore

$$\begin{aligned}&\dot{E}_0\Big(\Big\langle\exp\Big(-\int_0^t q(\dot{X}_s)ds\Big)\Big\rangle;\ \dot{X}_t=0\Big)\leqq\dot{E}_0\Big(\frac{1}{t}\int_0^t\langle e^{-tq(\dot{X}_s)}\rangle ds;\ \dot{X}_t=0\Big)\\&\qquad\leqq\langle e^{-tq(0)}\rangle=\int_{-\infty}^{\infty}e^{-t\lambda}dF(\lambda)\end{aligned}$$

applying Jensen inequality to

$$\exp\Big(-\int_0^t\{tq(\dot{X}_t)\}\cdot\frac{1}{t}\,ds\Big).$$

Combining (12) with Lemma 1 we obtain

$$e^{-\nu\sigma^2 t}\int_{-\infty}^{\infty} e^{-t\lambda}dF(\lambda)\leqq t\int_{-\infty}^{\infty} e^{-t\lambda}\rho(\lambda)d\lambda\leqq\int_{-\infty}^{\infty} e^{-t\lambda}dF(\lambda). \tag{13}$$

Our theorem is now immediate from (13) and Lemma 2. q.e.d.

The random difference operator $-H^0+q$ has been introduced by one of the authors [1] where $q(a)$, $a\in Z^\nu$, are assumed to be independent identically distributed non-negative random variables. Our theorem with $\alpha=2$ corresponds to a theorem of L. A. Pastur [2] in which the Schrödinger operator $-\Delta+q$ with $q(x)$, $x\in R^\nu$, being a stationary Gaussian random field is treated.

The reason why we call ρ the spectral distribution function is in that the following ergodic theorem holds. Let Λ be a rectangle containing the origin with sides parallel to axes. Let $\lambda_1^\omega\leqq\lambda_2^\omega\leqq\lambda_3^\omega\leqq\cdots\leqq\lambda_N^\omega$ be the eigenvalues of the problem: $(H^\omega u)(a)=\lambda u(a)$, $a\in\Lambda-\partial\Lambda$; $u(a)=0$, $a\in\partial\Lambda$. We put, for each λ, $\mathcal{N}^\omega(\lambda;\Lambda)=\sum_{\lambda_i^\omega\leqq\lambda}1$. Suppose the stationary random field $q(a)$, $a\in Z^\nu$, is metrically transitive, then there exists a set $\Omega_0\in\mathcal{B}$ with $P(\Omega_0)=1$ such that, for each $\omega\in\Omega_0$,

$$\lim_{\substack{L^{(i)}(\Lambda)\to\infty\\ i=1,2,\cdots,\nu}}\frac{\mathcal{N}^\omega(\lambda;\Lambda)}{|\Lambda|}=\rho(\lambda)$$

at every continuity point λ of $\rho(\lambda)$. Here $L^{(i)}(\Lambda)$ (resp. $|\Lambda|$) is the side length (resp. volume) of Λ.

The proof of Lemma 1, Lemma 2 and the above ergodic theorem will be given elsewhere.

References

[1] M. Fukushima: On the spectral distribution of a disordered system and the range of a random walk. Osaka Journal of Math., **11**, 73–85 (1974).

[2] L. A. Pastur: Spectra of random self-adjoint operator. Russian Mathematical Surveys, **28**, 1–67 (1973).