52. On Subclasses of Hyponormal Operators

By Masatoshi FUJII and Yasuhiko NAKATSU
Department of Mathematics, Osaka Kyoiku University

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1975)

1. We shall consider a (bounded linear) operator T acting on a Hilbert space $\mathfrak{H}$. An operator T is *hyponormal* if $TT^* \leqq T^*T$. And T is *quasinormal* if T commutes with T^*T. In [2] and [3], Campbell has discussed a subclass of hyponormal operators: An operator T is *heminormal* if T is hyponormal and T^*T commutes with TT^*. The subclass is called $(BN)^+$ in [3]. Also he proved

Theorem A. *If T is heminormal, then T^n is hyponormal for every n.*

We shall define a new class of operators to improve Theorem A. For each k, an operator T is *k-hyponormal* if

(1) $$(TT^*)^k \leqq (T^*T)^k.$$

Since $f(\lambda)=\lambda^\alpha$ for $0 \leqq \alpha \leqq 1$ is operator monotone, every k-hyponormal operator is hyponormal.

In this note, in § 2 we shall give characterizations of heminormal, quasinormal and k-hyponormal operators by means of an operator equation due to Douglas [4]. In § 3, we shall show that every heminormal operator is n-hyponormal for every n, and for each k, if T is k-hyponormal, then T^k is hyponormal.

2. In this section, we shall characterize heminormal, quasinormal and k-hyponormal operators. In [4], Douglas showed the following

Theorem B. *Let A and B be operators on $\mathfrak{H}$. Then $AA^* \leqq \lambda^2 BB^*$ for some $\lambda \geqq 0$ if and only if there is an operator C such that $A=BC$.*

In the proof of Theorem B, an operator C is constructed as follows; (i) $C^*(B^*x)=A^*x$ for every $x \in \mathfrak{H}$, (ii) C^* vanishes on $\operatorname{ran}(B^*)^\perp$, and (iii) $\|C\| \leqq \lambda$.

Now we shall give a characterization of heminormal operators.

Theorem 1. *An operator T is heminormal if and only if there is a positive contraction P such that*

(2) $$TT^* = PT^*T.$$

Proof. Suppose that T is heminormal. Since T^*T commutes with TT^*, we have $(TT^*)^2 \leqq (T^*T)^2$. It follows from Theorem B that there is an operator C such that $TT^*=T^*TC$, i.e., $TT^*=C^*T^*T$. So we put $P=C^*$, then we have by the above remarks (i) and (ii)

$$(P(x_1+x_2), x_1+x_2)=(Px_1, x_1) \geqq 0$$

for every $x_1 \in \overline{\operatorname{ran}(T^*T)}$ and $x_2 \in \operatorname{ran}(T^*T)^\perp$, that is, $C^* \geqq 0$. Since P

is contractive by the above remark (iii), there is a positive contraction P with (2).

Conversely, suppose that there is a positive contraction P with (2). Since P commutes with T^*T, we have

$$T^*TTT^* = T^*TPT^*T = P(T^*T)^2 = TT^*T^*T,$$

so that T^*T commutes with TT^*. Also we have

$$TT^* = PT^*T = (T^*T)^{1/2}P(T^*T)^{1/2} \leqq T^*T,$$

which completes the proof.

Next we shall characterize quasinormal operators.

Theorem 2. *An operator T is quasinormal if and only if there is a projection P with* (2).

Proof. Suppose that T is quasinormal and $T = V|T|$ is the polar decomposition of T. It follows from [1; Lemma 4.1] that $|T|$ commutes with V. Then we have

$$TT^* = V|T|^2V^* = VV^*T^*T,$$

so that $P = VV^*$ is a projection with (2).

Conversely, suppose that there is a projection P with (2). Since T^*T commutes with P, we have

$$T^*TTT^* = T^*TPT^*T = (PT^*T)^2 = (TT^*)^2.$$

Since $\operatorname{ran}(T^*)^\perp = \ker(T)$, we have $T^*T^2 = TT^*T$. Hence T is quasinormal.

Remark. If P is a positive operator with (2), then P commutes with TT^*. Actually, we have

$$PTT^* = P^2T^*T = PT^*TP = TT^*P.$$

We shall give a similar characterization for 2-hyponormal operators.

Theorem 3. *An operator T is* 2*-hyponormal if and only if there is a contraction P with* (2).

Proof. As in the proof of Theorem 1, for every 2-hyponormal operator, there is a contraction P with (2). If there is a contraction P with (2), then we have

$$(TT^*)^2 = (PT^*T)^*(PT^*T) = T^*TP^*PT^*T \leqq (T^*T)^2,$$

which completes the proof.

Remark. By a similar proof, we can show that T is k-hyponormal if and only if there is a contraction P such that $(TT^*)^{k/2} = P(T^*T)^{k/2}$.

3. In this section, we shall discuss on k-hyponormal operators. At first, we shall show the following

Theorem 4. *Every heminormal operator is k-hyponormal for every k.*

Proof. By the assumption, we have

$$\begin{aligned}(T^*T)^k - (TT^*)^k \\ &= (T^*T - TT^*)\{(T^*T)^{k-1} + (T^*T)^{k-2}(TT^*) + \cdots \\ &\quad + (T^*T)(TT^*)^{k-2} + (TT^*)^{k-1}\} \geqq 0.\end{aligned}$$

It is known that $f(\lambda)=\lambda^{\alpha}$ is operator monotone for every $0\leqq\alpha\leqq 1$, cf. [5]. Hence we have

Lemma 5. *If T is k-hyponormal, then T is n-hyponormal for every $1\leqq n\leqq k$.*

Theorem 6. *For each k, if T is k-hyponormal, then T^k is hyponormal.*

Proof. Note that T is 1-hyponormal if and only if T is hyponormal. We shall prove inductively that $T^kT^{*k}\leqq(TT^*)^k$ and $(T^*T)^k\leqq T^{*k}T^k$. Suppose that they are true for $k=n-1$ and T is n-hyponormal. Then we have by Lemma 5

$$T^nT^{*n}=T(T^{n-1}T^{*n-1})T^*\leqq T(TT^*)^{n-1}T^*\leqq T(T^*T)^{n-1}T^*=(TT^*)^n,$$

and

$$(T^*T)^n=T^*(TT^*)^{n-1}T\leqq T^*(T^*T)^{n-1}T\leqq T^*(T^{*n-1}T^{n-1})T=T^{*n}T^n.$$

Therefore, if T is k-hyponormal, then we have

$$T^kT^{*k}\leqq(TT^*)^k\leqq(T^*T)^k\leqq T^{*k}T^k,$$

which completes the proof.

By Theorem 4 and Theorem 6, we have the following theorem due to Campbell.

Theorem C ([3]). *If T is heminormal, then T^n is hyponormal for every n.*

It is known that if T is invertible and hyponormal, then T^{-1} is hyponormal. We have an analogous result on k-hyponormal operators.

Theorem 7. *For each k, if T is invertible and k-hyponormal, then T^{-1} is k-hyponormal.*

Proof. Since $A=(TT^*)^k$ and $B=(T^*T)^k$ are invertible and $0\leqq A\leqq B$, then we have $B^{-1/2}AB^{-1/2}\leqq 1$, so that $1\leqq B^{1/2}A^{-1}B^{1/2}$. Hence we have $B^{-1}\leqq A^{-1}$, that is,

$$(T^{-1}T^{*-1})^k=(T^*T)^{-k}\leqq(TT^*)^{-k}=(T^{*-1}T^{-1})^k.$$

4. A factorization of hyponormal operators is also discussed by T. Saito in his unpublished paper [6]. We obtain relations among subnormal, heminormal and k-hyponormal operators as follows:

(1) There is a hyponormal operator which is not k-hyponormal.

(2) There is a k-hyponormal operator which is not heminormal.

(3) There is a subnormal operator which is not k-hyponormal.

(4) There is a heminormal operator which is not subnormal. (4) is showed in [2]. The proofs will appear in a separate paper.

References

[1] A. Brown: On a class of operators. Proc. Amer. Math. Soc., **4**, 723–728 (1953).

[2] S. L. Campbell: Linear operators for which T^*T and TT^* commute. Proc. Amer. Math. Soc., **34**, 177–180 (1972).

[3] S. L. Campbell: Linear operators for which T^*T and TT^* commute. II. Pacific J. Math., **53**, 355–361 (1974).
[4] R. G. Douglas: On majorization, factorization and range inclusion of operators on Hilbert space. Proc. Amer. Math. Soc., **17**, 413–415 (1966).
[5] G. K. Pedersen: Some operator monotone functions. Proc. Amer. Math. Soc., **36**, 309–310 (1972).
[6] T. Saito: Factorization of hyponormal operators (to appear).