84. *Another Form of the Whitehead Theorem in Shape Theory*

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1. Introduction. In a previous paper [5] we have established the following theorem which is a shape-theoretical analogue of the classical Whitehead theorem in homotopy theory of CW complexes.

Theorem 1.1. *Let* $f:(X,x_0)\to(Y,y_0)$ *be a shape morphism of pointed connected topological spaces of finite dimension. If the induced morphisms* $\pi_k(f):\pi_k\{(X,x_0)\}\to\pi_k\{(Y,y_0)\}$ *of homotopy pro-groups*[1] *is an isomorphism for* $1\leqq k<n$ *and an epimorphism for* $k=n$ *where* $n=\max(1+\dim X, \dim Y)$, *then* f *is a shape equivalence.*

The purpose of this note is to prove the following theorem, which corresponds to another form of the Whitehead theorem in homotopy theory of CW complexes; Theorem 1.2 was announced in a previous paper [5].

Theorem 1.2. *Let* $f:(X,x_0)\to(Y,y_0)$ *be the same as in Theorem* 1.1. *If the induced morphism* $\pi_k(f):\pi_k\{(X,x_0)\}\to\pi_k\{(Y,y_0)\}$ *of homotopy pro-groups is an isomorphism for* $1\leqq k\leqq n$ *where* $n=\max(\dim X,\dim Y)$, *then* f *is a shape equivalence.*

Furthermore, the following theorem holds.

Theorem 1.3. *Let* $f:(X,x_0)\to(Y,y_0)$ *be a shape morphism of pointed connected topological spaces such that the induced morphism*

$$\pi_k(f):\pi_k\{(X,x_0)\}\longrightarrow\pi_k\{(Y,y_0)\}$$

of homotopy pro-groups is an isomorphism for $1\leqq k\leqq n$. *If* $\dim Y\leqq n$, *then there exists a unique shape morphism* $g:(Y,y_0)\to(X,x_0)$ *such that* $fg=1$.

2. Preliminaries. Let $f:(X,x_0)\to(Y,y_0)$ be a shape morphism of pointed connected topological spaces.

As in [5], without loss of generality we may assume that $\{(X_\lambda,x_{0\lambda}),[p_{\lambda\lambda'}],\Lambda\}$ and $\{(Y_\lambda,y_{0\lambda}),[q_{\lambda\lambda'}],\Lambda\}$ are inverse systems in $\mathfrak{W}_0$ which are isomorphic to the Čech systems of (X,x_0) and (Y,y_0) respectively in pro$(\mathfrak{W}_0)$, where $\mathfrak{W}_0$ is the homotopy category of pointed connected CW complexes, and that f is an equivalence class containing a special system map

$$\{1,f_\lambda,\Lambda\}:\{(X_\lambda,x_{0\lambda}),[p_{\lambda\lambda'}],\Lambda\}\longrightarrow\{(Y_\lambda,y_{0\lambda}),[q_{\lambda\lambda'}],\Lambda\}$$

1) For the definition of the k-th homotopy pro-group of a pointed topological space (X,x_0), see [5]. Here we denote it by $\pi_k\{(X,x_0)\}$ (cf. [2]).

with each f_λ a cellular map. Moreover, by [3] we can assume that $\dim X_\lambda \leqq \dim X$ and $\dim Y_\lambda \leqq \dim Y$ for each $\lambda \in \Lambda$.

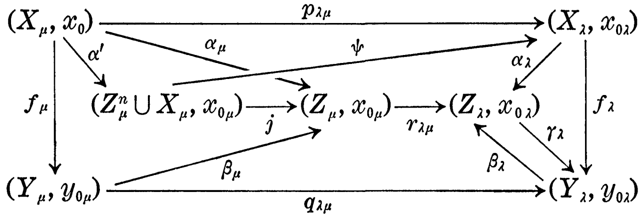
For each $\lambda \in \Lambda$, let Z_λ be the reduced mapping cylinder of f_λ, which is obtained from the disjoint union of $X_\lambda \times I$ and Y_λ by identifying $(x, 1)$ with $f_\lambda(x)$ for $x \in X_\lambda$ and by shrinking $(x_{0\lambda} \times I) \cup \{y_{0\lambda}\}$ to a point which is denoted also by $x_{0\lambda}$; the images of (x, t) with $x \in X_\lambda$, $t \in I$ and of $y \in Y_\lambda$ under the identification are denoted by $[x, t]$ and $[y]$ respectively. Let us define embeddings $\alpha_\lambda : (X_\lambda, x_{0\lambda}) \to (Z_\lambda, x_{0\lambda})$, $\beta_\lambda : (Y_\lambda, y_{0\lambda}) \to (Z_\lambda, x_{0\lambda})$ by $\alpha_\lambda(x) = [x, 0]$, $\beta_\lambda(y) = [y]$ and a map $\gamma_\lambda : (Z_\lambda, x_{0\lambda}) \to (Y_\lambda, y_{0\lambda})$ by $\gamma_\lambda[x, t] = [f_\lambda(x)]$, $\gamma_\lambda[y] = y$. Then $f_\lambda = \gamma_\lambda \alpha_\lambda$, $\gamma_\lambda \beta_\lambda = 1$ and $\beta_\lambda \gamma_\lambda \simeq 1$.

The following lemma is a direct consequence of a result established in the proof of [5, Theorem C].[2] Here we denote the n-skeleton of a CW complex K by K^n as usual.

Lemma 2.1. *Suppose that the induced morphism* $\pi_k(f) : \pi_k\{(X, x_0)\} \to \pi_k\{(Y, y_0)\}$ *of homotopy pro-groups is an isomorphism for* $1 \leqq k < n$ *and an epimorphism for* $k = n$. *Then for any* $\lambda \in \Lambda$ *there exist* $\mu \in \Lambda$ *with* $\lambda \leq \mu$ *and continuous maps*

$$\psi : (Z_\mu^n \cup X_\mu, x_{0\mu}) \longrightarrow (X_\lambda, x_{0\lambda}), \qquad r_{\lambda\mu} : (Z_\mu, x_{0\mu}) \longrightarrow (X_\lambda, x_{0\lambda})$$

such that the diagram



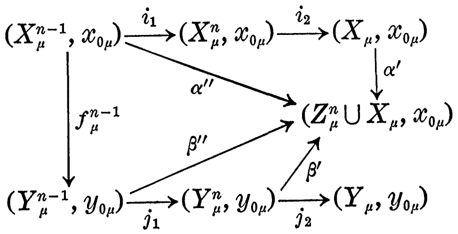
is homotopy commutative, where

$$\alpha' : (X_\mu, x_{0\mu}) \longrightarrow (Z_\mu^n \cup X_\mu, x_{0\mu}), \qquad j : (Z_\mu^n \cup X_\mu, x_{0\mu}) \longrightarrow (Z_\mu, x_{0\mu})$$

are inclusion maps.

Furthermore we have

Lemma 2.2. *Under the same assumption as in Lemma* 2.1, *the diagram*



is homotopy commutative and $f_\lambda \psi \beta' \simeq q_{\lambda\mu} j_2$, *where* i_1, i_2, j_1 *and* j_2 *are in-*

2) Correction to [5, p. 252]: line 20, for "$\beta_\lambda(E^1 \times I) = x_1$" read "$\beta_\lambda(E^1 \times 1) = x_1$"; line 25, for "$(E^k \times 0 \cup E^k \times I, E^k$" read "$(E^k \times 0 \cup \dot{E}^k \times I, \dot{E}^k$"; line 26, for "$(X_{k-1}, A_{k-1}, x_{k-1})$" read "$(X_{k-1}, A_{k-1}, x'_{k-1})$"; line 27, for "such that $h_\lambda(s_0) \in X_0^1$" read "and $x'_{k-1} = \alpha_\lambda(s_0, 1)$".

clusion maps and

$$\alpha''=\alpha_\mu \mid (X_\mu^{n-1}, x_{0\mu}) : (X_\mu^{n-1}, x_{0\mu}) \longrightarrow (Z_\mu^n \cup X_\mu, x_{0\mu}),$$
$$\beta'=\beta_\mu \mid (Y_\mu^n, y_{0\mu}) : (Y_\mu^n, y_{0\mu}) \longrightarrow (Z_\mu^n \cup X_\mu, x_{0\mu})$$

and $\beta''=\beta' j_1$.

Proof. The first part is obvious. We have $f_\lambda \psi \beta' \simeq \gamma_\lambda r_{\lambda\mu} j \beta' \simeq \gamma_\lambda r_{\lambda\mu} \beta_\mu j_2 \simeq q_{\lambda\mu} j_2$.

3. Proof of Theorem 1.2. We are now in a position to prove Theorem 1.2. Indeed, Theorem 1.2 is a direct consequence of Theorem 1.1, combined with Lemma 3.1 below.

Lemma 3.1. *Suppose that* $\pi_k(f) : \pi_k\{(X, x_0)\} \to \pi_k\{(Y, y_0)\}$ *is an isomorphism for* $1 \leqq k < n$ *and an epimorphism for* $k=n$. *If* $\dim Y \leqq n$, *then* $\pi_k(f) : \pi_k\{(X, x_0)\} \to \pi_k\{(Y, y_0)\}$ *is an epimorphism for* $k \geqq n$.

Proof. In this case, for any $\lambda \in \Lambda$ there exist some $\mu \in \Lambda$, a continuous map $\phi_{\lambda\mu} : (Y_\mu, y_{0\mu}) \to (X_\lambda, x_{0\lambda})$ such that

$$(1) \qquad q_{\lambda\mu} \simeq f_\lambda \phi_{\lambda\mu}.$$

This is seen from Lemma 2.2 by putting $\phi_{\lambda\mu}=\psi\beta'$. From (1) it follows that for any $k \geqq 1$ we have

$$(2) \qquad \pi_k(q_{\lambda\mu})=\pi_k(f_\lambda)\pi_k(\phi_{\lambda\mu}) : \pi_k(Y_\mu, y_{0\mu}) \longrightarrow \pi_k(Y_\lambda, y_{0\lambda}).$$

Hence we have

$$(3) \qquad \operatorname{Im} \pi_k(q_{\lambda\mu}) \subset \operatorname{Im} \pi_k(f_\lambda).$$

By [5, Theorem 1.2] (3) shows that

$$\pi_k(f) : \pi_k\{(X, x_0)\} \longrightarrow \pi_k\{(Y, y_0)\}$$

is an epimorphism in the category of pro-groups. This completes the proof.

4. Proof of Theorem 1.3. Let (P, p_0) and (Q, q_0) be pointed topological spaces. Let us denote by $[P, Q]$ the set of all the homotopy classes of continuous maps from (P, p_0) to (Q, q_0). The set of all shape morphisms from (P, p_0) to (Q, q_0) is denoted by $\mathfrak{S}_0[P, Q]$. Here for the sake of simplicity we omit the description of base-points in both case.

Let (Q, q_0) be a pointed CW complex, and Q^m the m-skeleton of Q. Then the inclusion map $i : (Q^m, q_0) \to (Q, q_0)$ induces a map

$$(i)_\# : [P, Q^m] \longrightarrow [P, Q].$$

Lemma 4.1. $(i)_\#$ *is surjective if* $\dim P \leqq m$ *and bijective if* $\dim P < m$.

Proof. Suppose that $\dim P \leqq m$ and that $g : (P, p_0) \to (Q, q_0)$ is a continuous map. Then by [3, Lemma 4.1] there exist a pointed CW complex (K, k_0) of dimension $\leqq m$ and two continuous maps $p : (P, p_0) \to (K, k_0)$, $\phi : (K, k_0) \to (Q, q_0)$ such that $g \simeq \phi p$. Here we may assume that ϕ is cellular. Hence, if we put $g'=\phi p$, then $[g]=(i)_\#[g']$, $[g'] \in [P, Q^m]$.

Next, suppose that $\dim P < m$, $[g_1], [g_2] \in [P, Q^m]$ and $(i)_\#[g_1] =(i)_\#[g_2]$. Then by [3, Lemmas 4.1 and 4.2] there exist a CW complex (K, k_0) of dimension $<m$ and continuous maps $p:(P, p_0)\to(K, k_0)$, $\phi_1, \phi_2:(K, k_0)\to(Q, q_0)$ such that $g_i\simeq\phi_i p$, $i=1, 2$ and $i\phi_1\simeq i\phi_2$. Here we may assume that ϕ_1 and ϕ_2 are cellular maps. Hence $\phi_1\simeq\phi_2:(K, k_0)\to(Q^m, q_0)$. Hence $[g_1]=[g_2]$. This proves Lemma 4.1.

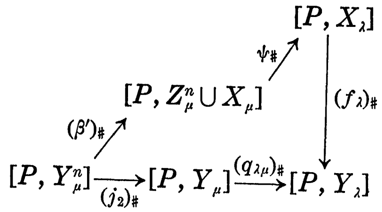
Now, let $f:(X, x_0)\to(Y, y_0)$ be a shape morphism such that $\pi_k\{(X, x_0)\}\to\pi_k\{(Y, y_0)\}$ is an isomorphism for $1\leqq k<n$ and an epimorphism for $k=n$. Let $\{(X_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\}$, $\{(Y_\lambda, y_{0\lambda}), [q_{\lambda\lambda'}], \Lambda\}$ and

$$\{1, f_\lambda, \Lambda\}:\{(X_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\}\longrightarrow\{(Y_\lambda, y_{0\lambda}), [q_{\lambda\lambda'}]\cdot\Lambda\}$$

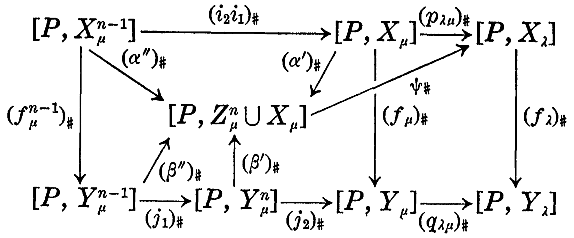
be the same as in § 2. We have Lemma 4.2 below by Lemma 2.1.

Lemma 4.2. *Let $\lambda, \mu\in\Lambda$ be the same as in Lemma 2.1. Then we have the homotopy commutative diagrams below:*

(i) *in case* $\dim P\leqq n$:



(ii) *in case* $\dim P<n$:



Now, let us first treat the case $\dim P\leqq n$. In this case, by Lemma 4.1 the map $(j_2)_\#$ in the first diagram of Lemma 4.2 is surjective. Hence we have

(4) $$\operatorname{Im}(q_{\lambda\mu})_\#\subset\operatorname{Im}(f_\lambda)_\#.$$

It is easy to see from (4) that the special system map

(5) $$\{1, (f_\lambda)_\#, \Lambda\}:\{[P, X_\lambda], (p_{\lambda\lambda'})_\#, \Lambda\}\longrightarrow\{[P, Y_\lambda], (q_{\lambda\lambda'})_\#, \Lambda\}$$

is an epimorphism in the pro-category of the category of sets.

Next, let us proceed to the case where $\dim P\leqq n-1$. In this case, by Lemma 4.1 $(j_2)_\#$ is bijective. Let us put

$$\phi_{\lambda\mu}=\psi_\#(\beta')_\#(j_2)_\#^{-2}:[P, Y_\mu]\longrightarrow[P, X_\lambda].$$

Then by Lemma 4.2 we have

(6) $$(f_\lambda)_\#\phi_{\lambda\mu}=(q_{\lambda\mu})_\#.$$

Let $[g]\in[P, X_\mu]$. Then by Lemma 4.1 there is $[h]\in[P, X^{n-1}_\mu]$ such that $[g]=(i_2i_1)_\#[h]$. On the other hand, by Lemma 4.2 we have $(\beta')_\#(j_1)_\#(f^{n-1}_\mu)_\# =(\beta'')_\#(f^{n-1}_\mu)_\#=(\alpha'')_\#$. Hence we have $\phi_{\lambda\mu}(f_\mu)_\#[g]=\phi_{\lambda\mu}(j_2)_\#(j_1)_\#(f^{n-1}_\mu)_\#[h] =\psi_\#(\alpha'')_\#[h]=(p_{\lambda\mu})_\#(i_2i_1)_\#[h]=(p_{\lambda u})_\#[g]$, that is,

$$\phi_{\lambda\mu}(f_\mu)_\# = (p_{\lambda\mu})_\#. \tag{7}$$

By [5, Theorem 1.1] it follows from (6) and (7) that the special system map (5) is an isomorphism in the pro-category of the category of sets.

By [4, Theorem 2.3] we may assume that

$$\mathfrak{S}_0[P, X] = \varprojlim \{[P, X_\lambda], (p_{\lambda\lambda'})_\#, \Lambda\},$$
$$\mathfrak{S}_0[P, Y] = \varprojlim \{[P, Y_\lambda], (q_{\lambda\lambda'})_\#, \Lambda\}.$$

Therefore, we have

Theorem 4.3. *Let $f:(X, x_0) \to (Y, y_0)$ be a shape morphism of pointed connected topological spaces such that $\pi_k(f): \pi_k\{(X, x_0)\} \to \pi_k\{(Y, y_0)\}$ is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k = n$. If (P, p_0) is a pointed space with $\dim P < n$, then the map*

$$f_\# : \mathfrak{S}[P, X] \longrightarrow \mathfrak{S}_0[P, Y]$$

induced by f is bijective.

Now, it is clear that Theorem 1.3 is a direct consequence of Lemma 3.1 and Theorem 4.2.

Addendum. Our results in this paper were obtained in August, 1974. Quite recently, J. Dydak [1] has introduced the notion of the deformation dimension of a topological space X, ddim X in notation, and proved that ddim $X \leqq n$ if and only if the Čech system of X is isomorphic to an inverse system of *CW* complexes of dimension $\leqq n$ in pro $(\mathfrak{W}_0)$. Thus, as is pointed out by him, our proof of Theorem 1.1 in [5] remains valid if "dim" is replaced by "ddim". He proved also Theorem 4.3 with "dim" replaced by "ddim"; his proof is different from ours but relies upon our Lemma 2.1 as well. Finally, we note that Theorems 1.2 and 1.3 remain true if "dim" is replaced by "ddim".

References

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