76. *Continuity of Homomorphism of Lie Algebras of Vector Fields*

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1. Introduction. For any smooth manifold M, let $\mathcal{A}(M)$ be the (infinite dimensional) Lie algebra formed by all the smooth vector fields on M under the usual bracket operation and Diff (M) the group formed by all the diffeomorphisms of M. In [3] (Theorem 1.3.2) H. Omori proved that if M and N are compact and $\varphi: \mathcal{A}(M) \to \mathcal{A}(N)$ is a Lie algebra homomorphism which is continuous in the C^∞-topology, then φ induces a local homomorphism Diff $(M) \to$ Diff (N) as in the finite dimensional case. In this theorem the assumption of the continuity can be omitted, i.e. we can prove the following

Theorem. *Any homomorphism $\varphi: \mathcal{A}(M) \to \mathcal{A}(N)$ is continuous in the C^∞-topology.*

Since it can be shown that if φ is non-trivial and N is compact then M is also compact, we have

Corollary. *If N is compact then φ induces a local homomorphism* Diff $(M) \to$ Diff (N).

It is known that if φ is an isomorphism, then M and N are diffeomorphic, in other words, the Lie algebra $\mathcal{A}(M)$ determines the manifold M ([4], for non-compact case [2]). In case of the general homomorphism, the relation of M and N is given as follows. For any positive integer l, let M_l be a smooth manifold formed by all the sets of distinct l points of M and put $N_0 = \{q \in N \mid \varphi(X)$ vanishes at q for any $X \in \mathcal{A}(M)\}$. Then N is a finite disjoint union of subsets $N_0, N_1, \cdots, N_k$ and if N is compact then each N_l is a (topological) fibre bundle over M_l. This bundle is closely related to the jet bundle of the tangent bundle of $M^l = M \times \cdots \times M$. (It seems that $N_0 = \phi$ and N_l is a smooth bundle whose fibre is a smooth manifold with corner.) The details will appear elsewhere.

2. Sketch of the proof of Theorem. Recall that the C^∞-topology of $\mathcal{A}(M)$ is given by seminorms $|\cdot|_{U,r}$ defined as follows. Let U be a relatively compact open set of M and $(x) = (x^1, \cdots, x^n)$ a coordinate system on some neighborhood of $\overline{U}$. Then for any $X \in \mathcal{A}(M)$ with $X = \sum f^i(x)\partial_{x^i}$ on U, we put

$$|X|_{U,r} = \sup_{x \in U, |\alpha| \leq r, i \leq n} |D^\alpha f^i(x)|$$

where ∂_{x^i} and D^α denote the vector field $\partial/\partial x^i$ and the differential operator $\partial^{|\alpha|}/(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}$ respectively where $|\alpha|=\alpha_1+\cdots+\alpha_n$ for any multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$. To prove the continuity of φ, we shall express φ in terms of coordinate systems of M and N. For this purpose we need the following theorem, essentially due to I. Amemiya [1]. For any point p of M, put $\mathcal{M}_p=\{f\in C^\infty(M)\mid f(p)=0\}$.

Theorem 1. *Let $\mathcal{B}$ be a proper subalgebra of $\mathcal{A}(M)$ with* $\operatorname{codim}\mathcal{B}=d<\infty$. *Then we can find a finite number of points $p_1,\cdots,p_l$ of M such that*

$$\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}\mathcal{A}(M)\supset\mathcal{B}\supset\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M)$$

where $h=2((d-nl)^2+d-nl)+1$ and $n=\dim M$. Moreover we have $l\leqq d/n$.

For any $q\in N-N_0$ we have $0<\operatorname{codim}\varphi^{-1}\mathcal{M}_q\mathcal{A}(N)\leqq\operatorname{codim}\mathcal{M}_q\mathcal{A}(N)=\dim N<\infty$, hence by Theorem 1,

$$\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}\mathcal{A}(M)\supset\mathcal{B}\supset\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M) \tag{1}$$

holds for some $p_1,\cdots,p_l$ of M. Note that the set $\{p_1,\cdots,p_l\}$ is uniquely determined by (1). We denote by ψ the map which corresponds the set $\{p_1,\cdots,p_l\}$ to the point q of $N-N_0$. For each integer l, let N_l be the set of points q of $N-N_0$ such that the number of the corresponding p_ν's is l. We can show that if N is compact then N_l is a fibre bundle over M_l with the projection map ψ. Now, it follows easily from (1) that if X and Y have the same h-jets at $p_1,\cdots,p_l$ then $\varphi(X)=\varphi(Y)$ at q. Therefore if $X=\sum f_\nu^i(x_\nu)\partial_{x_\nu^i}$ on some neighborhood of p_ν for each ν, then the value of $\varphi(X)$ at q is given by $\sum D^\alpha f_\nu^i(p_\nu)Z_{i\nu}^\alpha$ for some vectors Z. By some calculations we can prove the next

Theorem 2. i) *There exists an open subset N_l^+ of* $\operatorname{Int}N_l$ *for each l such that $\bigcup N_l^+$ is dense in $N-N_0=\bigcup N_l$.*

ii) *Let q be a point of N_l^+ with $\psi(q)=\{p_1,\cdots,p_l\}$ and $(x_\nu)=(x_\nu^1,\cdots,x_\nu^n)$ be a coordinate system on some neighborhood U_ν of p_ν for each ν. Then there exists a coordinate system $(x_*,y)=(x_1,\cdots,x_l,y)=(x_1^1,\cdots,x_1^n,\cdots,x_l^n,y^1,\cdots,y^{d-nl})$ on some neighborhood U of q satisfying the following.*

a) $\psi(x_*,y)=\{(x_1),\cdots,(x_l)\}$.

b) *For any $X\in\mathcal{A}(M)$ with $X=\sum_i f_\nu^i(x_\nu)\partial_{x_\nu^i}$ on each U_ν we have*

$$\varphi(X)(x_*,y)=\sum_\nu\sum_i f_\nu^i(x_\nu)\partial_{x_\nu^i}+\sum_{0<|\alpha|\leqq h}\frac{D^\alpha}{\alpha!}f_\nu^i(x_\nu)Y_{i\nu}^\alpha(y)$$

on U where $h=2((d-nl)^2+d-nl)+1$, $n=\dim M$, $d=\dim N$ and $Y_{i\nu}^\alpha(y)=\sum_j Y_{i\nu}^{\alpha j}(y)\partial_{y^j}$.

c) *Y's satisfy*

$[Y_{i\nu}^\alpha, Y_{j\mu}^\beta]=0$ *for $\nu\neq\mu$ and* $[Y_{i\nu}^\alpha, Y_{j\nu}^\beta]=\beta_i Y_{j\nu}^{\alpha+\beta-i}-\alpha_j Y_{i\nu}^{\alpha+\beta-j}$.

(Note that these relations are the same as $x_\nu^\alpha\partial_{x_\nu^i}$'s satisfy.)

Let $(v)=(v^1, \cdots, v^d)$ be a coordinate system on some open set V of N and put $\varphi(X)=\sum \varphi^p(X)(v)\partial_{vp}$ on V. To prove the continuity of φ, we must estimate $D^\beta\varphi^p(X)$ for $|\beta|\leqq r$, which equals, by Theorem 2, $\sum_{\nu,i}\sum_{|\gamma|\leqq h+r} D^\gamma f^i_\nu(x)Z^{\gamma\beta p}_{i\nu}(v)$ on $U\cap V$ where Z's are smooth functions which are not necessarily bounded on $U\cap V$. We use the following lemma to eliminate Z.

Lemma. *Let $\Phi: C^\infty(\boldsymbol{R}^n)\to C^\infty(\boldsymbol{R}^{nl})[Z^\alpha_\nu]$ (=the polynomial ring with $C^\infty(\boldsymbol{R}^{nl})$ coefficient) be a map given by*

$$\Phi(f(x))=\sum_{\nu=1}^{l}\sum_{|\alpha|\leqq h} D^\alpha f(x_\nu)Z^\alpha_\nu.$$

Then we have

$$\Phi(f(x))=f(x_1)\Phi(1)+\sum_{k=1}^{l(h+1)-1}\sum_{j_1,\cdots,j_k=1}^{n}\int_0^1\cdots\int_0^1 \partial_{j_1}\cdots\partial_{j_k}f(x(k))dt(k)$$
$$\times\sum_{m=0}^{k}(-1)^m\sum_{1\leqq i_1<\cdots<i_m\leqq k} x^{j_1}_{i_1}\cdots x^{j_m}_{i_m}\Phi(x^{j_{m+1}+\cdots+j_k})\Big|_{\substack{x_\nu+l s=x_\nu\\ \nu\leqq l,s\leqq h}},$$

where

$$x(k)=(1-t_1)x_1+(1-t_2)t_1x_2+\cdots+(1-t_k)t_{k-1}\cdots t_1x_k+t_kt_{k-1}\cdots t_1x_{k+1},$$
$$dt(k)=t_1^{k-1}t_2^{k-2}\cdots t_{k-1}dt_1\cdots dt_k.$$

Applying this lemma to $\Phi(f(x))=\sum_{\nu=1}^{l}\sum_{|\gamma|\leqq h+r}D^\gamma f(x_\nu)Z^{\gamma\beta p}_{i\nu}(v)$ for each $i\leqq n$, we obtain

$$|D^\beta\varphi^p(X)(v)|\leqq C|X|_{W,ar+b}$$

on $U\cap V$ for some constant C where W is a suitable open set of M, $a=[d/n]$=the integer part of d/n and $b=2a((d-n)^2+d-n+1)-1$. Using this inequality we can prove the continuity of φ.

References

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