57. A Sharp Form of the Existence Theorem for Hyperbolic Mixed Problems of Second Order

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§1. **Introduction.** In this paper we consider the following initial boundary value problem

$$\{P, B\}\begin{cases} Pu=f(x,t), & \text{for } x\in\Omega,\ t>0,\\ Bu\big|_{\partial\Omega}=g(s,t) & \text{for } s\ni\partial\Omega,\ t>0,\\ D_t^j u\big|_{t=0}=u_j(x),\ (j=0,1), & \text{for } x\in\Omega,\end{cases}$$

in the cylindrical domain $\Omega\times(0,\infty)$, where Ω is the exterior or the interior of a smooth and compact hypersurface $\partial\Omega$ in R^{n+1}. P is a regularly hyperbolic operator with respect to t, and $\partial\Omega$ is non-characteristic to P. Moreover we assume that the only one of $\tau_1(\nu)$ and $\tau_2(\nu)$ is negative for all $(s,t)\in\partial\Omega\times(0,\infty)$, where $\tau_j(\xi)$ are the roots of $P(s,t;\xi,\tau)=0$ and ν is the inner unit normal at (s,t). This condition means that the number of boundary conditions is one. B is a first order operator:

$$B=B(s,t;D_x,D_t)=\sum_{j=1}^{n+1} b_j(s,t)D_{x_j}-c(s,t)D_t,\qquad D_t=\frac{1}{i}\frac{\partial}{\partial t}\qquad\text{etc.},$$

where $\sum_{j=1}^{n+1} b_j(s,t)\nu_j=B(s,t,\nu,0)=1$. We assume that all the coefficients are smooth and bounded, and that they remain constant outside some compact sets.

We are concerned with the following question: Under what condition the solution $u(t)$ of $\{P,B\}$ has the continuity for the initial data in the same Sobolev space? The answer is just the condition (H) below, which was derived in [2].[1] We state it as

Theorem 1. *The necessary and sufficient condition that the energy inequality*

1) (H) was introduced as a characterization of problems which satisfy

$$\gamma|u|_{1,\gamma}^2\leqq\frac{c}{\gamma}|Pu|_{0,\gamma}^2,$$

holds for any smooth function with compact support satisfying the homogeneous boundary condition, in the case of constant coefficients. See also [1] and [3]. In [2] we proved the existence theorem with the initial data in a weaker sense. It is difficult to prove the estimate (1.1) as the direct extension of the arguments in [2]. For this purpose we need more precise considerations on the global properties of (H).

$$(1.1) \qquad \sum_{j=0}^{1} \|(D_t^j u)(t)\|_{1-j} \le C(T)\left\{\sum_{j=0}^{1} \|(D_t^j u)(0)\|_{1-j} + \int_0^t \|(Pu)(s)\|_0\, ds\right\}$$

holds for any $u=u(x,t)$ *in* $C_0^\infty(\bar{\Omega}\times R^1)$ *satisfying* $Bu\,\big|_{\partial\Omega}=0$ *and for any* t *in* $(0,T)$ *with some constant* $C(T)$, *is the following condition* (H).

(H): *For all* $(s,t,\eta)\in\partial\Omega\times R^1\times S^n$, $(\eta\cdot\nu=0)$, $\{P,B\}$ *satisfies the followings.* (*We state the case* $P=\Box$. *In general, see Theorem* 2 *in* [2].)

(I) $A=\begin{pmatrix} 2\,\mathrm{Re}\,\alpha & \mathrm{Im}\,(\alpha\bar{\beta}) \\ \mathrm{Im}\,(\alpha\bar{\beta}) & 2\,\mathrm{Re}\,\beta \end{pmatrix}\ge 0$, *when* $|\mathrm{Re}\,\alpha|+|\mathrm{Re}\,\beta|\ne 0$,

(II) $1+(\mathrm{Im}\,\alpha)(\mathrm{Im}\,\beta)\ge\delta>0$, *when* $|\mathrm{Re}\,\alpha|+|\mathrm{Re}\,\beta|=0$.

Here $\alpha=c(s,t)+B(s,t,\eta,0)/|\eta|$ *and* $\beta=c(s,t)-B(s,t,\eta,0)/|\eta|$.

$\|u(t)\|_k$ means Sobolev k-norm in Ω.

We can say more, that is

Theorem 2. *Suppose* (H). *If* $f(t)\in\mathcal{E}_t^0(L^2(\Omega))$,[2] $g\in\mathcal{E}_t^0(H^{\frac{1}{2}}(\partial\Omega))$ *and* $u_j\in H^{1-j}(\Omega)$, $(j=0,1)$, *then there exists a solution* $u(t)$ *of* $\{P,B\}$ *in* $\mathcal{E}_t^0(H^1(\Omega))\cap\mathcal{E}_t^1(L_2(\Omega))$ *satisfying the following energy estimate* (*E*) *with* $k=0$. *Moreover if we assume that the smooth data* $\{f,g,u_0,u_1\}$ *satisfy the compatibility condition*[3] *of order* k, $(k\ge 1)$, *then the solution satisfies*

$$(E)\ \sum_{j=0}^{1}\|(D_t^j u)(t)\|_{1-j+k}^2+\gamma\sum_{i+j\le k}\int_0^t e^{\gamma(t-s)}\langle\!\langle (D_x^i D_t^j u(s))\rangle\!\rangle_{\frac{1}{2}-i-j+k}^2 ds$$
$$\le Ce^{\gamma t}\left\{\sum_{j=0}^{1}\|u_j\|_{1-j+k}^2+\frac{1}{\gamma}\sum_{j\le k}\int_0^t e^{-\gamma s}\big(\|(D_t^j f)(s)\|_{k-j}^2+\langle\!\langle D_t^j g)(s)\rangle\!\rangle_{\frac{1}{2}+k-j}^2\big)ds\right\},$$

for $\gamma>\gamma_k$, *where* C *and* γ_k *are positive constants.*

$\langle\!\langle v(s)\rangle\!\rangle_r$ *means Sobolev* r*-norm in* $\partial\Omega$. *The solution has the same propagation speed as that in the case of Cauchy problem.*

Remark. If $g\equiv 0$ in the problem $\{P,B\}$, the above solution u satisfies (1.1).

The detailed proof will be given in a forthcoming paper. Here we sketch the proofs of (1.1) and (*E*) in the case where $P=\Box$ and $\Omega=R_+^{n+1}=\{(x,y): x>0, y\in R^n\}$ for simplicity.

§ 2. The choice of *Q*. We prove (1.1) by the integration by parts of

$$\mathcal{G}((0,t),P,Q;\varphi_j u)=2i\,\mathrm{Im}\int_0^t\int_{R_+^{n+1}} e^{-2\gamma t}P\varphi_j u\,\overline{Q\varphi_j u}\,dxdydt,$$

where Q is a suitable first order operator. Here $\varphi_j u=\overline{\mathcal{F}}_y\varphi_j\mathcal{F}_y u$ is a localization of u corresponding to the partition of unity:

$$(*) \qquad \sum_{j=0}^{\text{finite}}\varphi_j(x,y,t,\eta)\equiv 1 \quad \text{on} \quad \bar{R}_+^1\times R^n\times\bar{R}_+^1\times R^n,$$

where φ_j, $(j\ge 1)$, are homogeneous of degree zero in η for $|\eta|\ge 1$. We take Q in a neighbourhood of the boundary as follows:

2) $\mathcal{E}_t^k(H)\ni u(t)$ means that u is a continuous function in H, up to their k-th derivatives.

3) See § 9 in [2].

(I) $Q=(\alpha_1 z_1+\beta_1 z_2)+\varepsilon(z_1+cz_2-d\xi)$, if the supports of φ_j contain any point satisfying $\alpha_1\beta_1=0$, $(z_1=\tau-|\eta|, z_2=\tau+|\eta|, \alpha=\alpha_1+i\alpha_2, \beta=\beta_1+i\beta_2)$.
(II) $Q=(\alpha_1 z_1+\beta_1 z_2)-\varepsilon(2\xi-c_1 z_1-c_2 z_2)$, in other cases.
Here ε is a sufficiently small positive number and c_1, c_2 and c are positive functions in (y, t, η). We choose these as follows:

(1.1) $$2(\alpha_1 z+\beta_1)(c_1 z+c_2)=|\alpha|^2 z^2+2(2+\mathrm{Re}\,\alpha\bar{\beta})z+|\beta|^2+(\det A)/\alpha_1^2,$$

(1.2) $$\varepsilon_1<c<1/\varepsilon_2$$

(1.3) $$d=4(d_1X+d_2Y),$$

where $X=(\beta_1-\alpha_1\varepsilon_1)/\rho(1-\varepsilon_1\varepsilon_2)$,

$$Y=(\alpha_1-\beta_1\varepsilon_2)/\rho(1-\varepsilon_1\varepsilon_2),$$

and

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}=\frac{1}{1-\varepsilon_1\varepsilon_2}\begin{pmatrix} 1 & -\varepsilon_2 \\ -\varepsilon_1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix}.$$

Here ε_1, ε_2 and ρ are defined by

(1.4) $$|\alpha|^2 z^2+2(2+\mathrm{Re}\,\alpha\bar{\beta})z+|\beta|^2=\rho(z+\varepsilon_1)(1+\varepsilon_2 z).$$

We remark that for the estimates (1.1) and (E) we need the localization of type $(*)$. This makes the choice of Q difficult. In the actual calculations we employ a special device concerning the reverse process of Green formula which will be explained below. We need these, because the estimates (1.1) and (E) are finer than the estimate; $\gamma|u|_{1,\gamma}^2 \leq \frac{c}{\gamma}|Pu|_{0,\gamma}^2$ which was treated earlier in [2], [1] and [3].

§3. **Green formula associated with the boundary condition.** To $\mathscr{G}((0,t), P, Q; u)$ there corresponds the following symbolic calculus:

$$G(P,Q)=P(\xi,\eta,\tau)Q(\zeta,\eta,\bar{\tau})-Q(\xi,\eta,\tau)P(\zeta,\eta,\bar{\tau})$$
$$=(\xi-\zeta)G_x(P,Q)-(\tau-\bar{\tau})G_t(P,Q).$$

Here $G_x(P,Q)$ and $G_t(P,Q)$ are quadratic forms in (ξ, z_1, z_2), z_1 and z_2 being $z_1=\tau-|\eta|$ and $z_2=\tau+|\eta|$ respectively. Now, taking account of the boundary condition $D_x u\big|_{x=0}=\frac{1}{2}(\alpha z_1+\beta z_2)(D)u\big|_{x=0}+g$, we substitute $\frac{1}{2}(\alpha z_1+\beta z_2)$ into ξ in $G_x(P,Q)$, and $\frac{1}{2}(\bar{\alpha}\bar{z}_1+\bar{\beta}\bar{z}_2)$ into ζ in $G_x(P,Q)$. Then $G_x(P,Q)$ becomes an Hermite form $G'_x(P,Q)$ in (z_1, z_2). Denote the *anti-symmetric part* of $G'_x(P,Q)$ by $i\,\mathrm{Im}\,g(P,Q)_{1,2}(z_1\bar{z}_2-z_2\bar{z}_1)$ and notice that

$$(\xi-\zeta)\{z_1\bar{z}_2-z_2\bar{z}_1\}=-(\tau-\bar{\tau})\{\xi\bar{z}_1-\xi\bar{z}_2-z_1\zeta+z_2\xi\}.$$

Hence we have

$$G(P,Q)=(\xi-\zeta)\tilde{G}_x(P,Q)-(\tau-\bar{\tau})\tilde{G}_t(P,Q),$$

where $\tilde{G}_x(P,Q)$ is a symmetric part of $G'_x(P,Q)$ and $\tilde{G}_t$ is an *Hermite form*:

$$\tilde{G}_t(P,Q)=G_t(P,Q)+i\,\mathrm{Im}\,g(P,Q)_{1,2}\{\xi\bar{z}_1-\xi\bar{z}_2-z_1\zeta+z_2\zeta\}.$$

Using Q defined in §2 we can prove $\tilde{G}_x\geq 0$ and $\tilde{G}_t>0$.

References

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