91. *The Existence and Uniqueness of the Solution of Equations Describing Compressible Viscous Fluid Flow in a Domain*

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1. Introduction. The compressible viscous isotropic Newtonian fluid motion is described as follows: (the summation convention is used)

$$\frac{D\rho}{Dt}=-\rho\frac{\partial v_k}{\partial x_k}, \tag{1.1}$$

$$\begin{aligned}\frac{Dv_i}{Dt}=&\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\mu'\frac{\partial v_k}{\partial x_k}\right)+\frac{1}{\rho}\frac{\partial}{\partial x_k}\left[\mu\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)\right]\\&-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+f_i\qquad (i=1,2,3),\end{aligned} \tag{1.2}$$

$$\frac{DS}{Dt}=\frac{1}{\rho\theta}\frac{\partial}{\partial x_k}\left(\kappa\frac{\partial\theta}{\partial x_k}\right)+\frac{\mu}{2\rho\theta}\left(\frac{\partial v_j}{\partial x_k}+\frac{\partial v_k}{\partial x_j}\right)^2+\frac{\mu'}{\rho\theta}\left(\frac{\partial v_k}{\partial x_k}\right)^2, \tag{1.3}$$

(ρ, density; v, velocity; μ, coefficient of viscosity; μ', second coefficient of viscosity; κ, coefficient of heat conduction; p, pressure; f, outer force; S, entropy; θ, absolute temperature; $D/Dt=\partial/\partial t+v_k\cdot\partial/\partial x_k$).

By the physical requirements, μ, μ', κ, p and S are considered to be functions of ρ and θ such that

$$\mu'+\frac{2}{3}\mu\geqq 0;\ \mu,\kappa,p,S_\theta>0. \tag{1.4}$$

If S is smooth, then it follows from (1.1) and (1.3) that

$$\begin{aligned}\frac{D\theta}{Dt}=&\frac{1}{\rho\theta S_\theta}\frac{\partial}{\partial x_k}\left(\kappa\frac{\partial\theta}{\partial x_k}\right)+\frac{\mu}{2\rho\theta S_\theta}\left(\frac{\partial v_j}{\partial x_k}+\frac{\partial v_k}{\partial x_j}\right)^2\\&+\frac{\mu'}{\rho\theta S_\theta}\left(\frac{\partial v_k}{\partial x_k}\right)^2+\frac{\rho S_\rho}{S_\theta}\frac{\partial v_k}{\partial x_k}.\end{aligned} \tag{1.3$'$}$$

We shall consider a first initial-boundary value problem of (1.1), (1.2) and (1.3′) with the initial-boundary conditions:

$$\begin{cases}v(x,0)=v_0(x),\ \theta(x,0)=\theta_0(x),\ \rho(x,0)=\rho_0(x) & (x\in\Omega),\\ v(x,t)=0,\ \theta(x,t)=\theta_1(x,t) & ((x,t)\in\Gamma_T),\end{cases} \tag{1.5}$$

(Ω is a bounded or unbounded domain in R^3, whose boundary Γ belongs to $C^{2+\alpha}$ and satisfies Lyapunov conditions (cf. [4]); $\Gamma_T=\Gamma\times[0,T]$). We assume that the compatibility conditions hold and that in (1.5)

$$\begin{cases}v_0,\theta_0\in H^{2+\alpha}(\bar\Omega),\ \rho_0\in H^{1+\alpha}(\bar\Omega),\ 0<\bar\rho_0\leqq\rho_0\leqq\bar{\bar\rho}_0<+\infty,\\ 0<\bar\theta_0\leqq\theta_0\leqq\bar{\bar\theta}_0<+\infty,\ \theta_1\in H^{2+\alpha}(\Gamma_T),\ \mu,\mu',\kappa,p,S\in\mathcal{O}^{2+L}_{\mathrm{loc}}(\mathcal{D}_{\rho,\theta}),\\ f\in B^{1+L}(\bar Q_T),\end{cases} \tag{1.6}$$

$\Big(\mathcal{D}_{\rho,\theta}=\{(\rho,\theta)\in(0,\infty)\times(0,\infty)\}$;

$\bar{Q}_T=\bar{\Omega}\times[0,T]$;

$B^{n+\alpha}(\bar{Q}_T)=\Big\{g(x,t)\,\Big|\,\sum_{r+|s|=0}^{n}|D_t^rD_x^sg|_T^{(0)}+\sum_{r+|s|=n}|D_t^rD_x^sg|_T^{(\alpha)}<+\infty\Big\}$;

$$H^{n+\alpha}(\bar{Q}_T)=\Big\{v(x,t)\,\Big|\,\|v\|_T^{(n+\alpha)}\equiv\sum_{2r+|s|=0}^{n}|D_t^rD_x^sv|_T^{(0)} + \sum_{2r+|s|=(n-1)\vee 0}^{n}|D_t^rD_x^sv|_{t,T}^{(\alpha/2)}+\sum_{2r+|s|=n}|D_t^rD_x^sv|_{x,T}^{(\alpha)}<+\infty\Big\};$$

$|v|_T^{(0)}\equiv\sup_{\bar{Q}_T}|v|$;

$|v|_{t,T}^{(\alpha/2)}\equiv\sup\dfrac{|v(x,t)-v(x,t')|}{|t-t'|^{\alpha/2}}$;

$|v|_{x,T}^{(\alpha)}\equiv\sup\dfrac{|v(x,t)-v(x',t)|}{|x-x'|^{\alpha}}$;

$|v_{\Delta}^{(\alpha)}|\equiv|v|_{t,T}^{(\alpha/2)}+|v|_{x,T}^{(\alpha)}$;

when $\alpha=1$, notations such as $|v|_{x,T}^{(L)}$ are used; $\mathcal{O}_{\mathrm{loc}}^{n+L}(\mathcal{D}_{\rho,\theta})=\{q(\rho,\theta)\,|\,q$ is defined on $\mathcal{D}_{\rho,\theta}$, n-times partially differentiable and its n-th order derivatives are locally Lipschitz-continuous there$\}\Big)$.

Firstly we consider a characteristic curve $\bar{x}(\tau;x,t)$ of (1.1) passing (x,t) and put $x_0(x,t)=\bar{x}(0;x,t)$. If $v\in H^{2+\alpha}(\bar{Q}_T)$ with $v|_{\Gamma_T}=0$, then the correspondence $(x,t)\mapsto(x_0(x,t),t_0=t)$ is 1-to-1 from $\bar{Q}_T$ onto $\bar{Q}_T$ and the notation $(x(x_0,t_0),t=t_0)$ is used for the inverse transformation. Thus we have $x=x_0+\int_0^t\hat{v}(x_0,\tau)d\tau$, where $\hat{v}(x_0,t_0)=v(x(x_0,t_0),\ t=t_0)$, and we use these notations for other functions without explicit statements from now on. Denoting the inverse matrix $\left(\dfrac{\partial x}{\partial x_0}\right)^{-1}$ by (g_{ij}), according to (1.1) we have ((x,t) is used in place of (x_0,t_0) for simplicity)

$$\hat{\rho}(x,t;\hat{v})=\rho_0(x)\cdot\exp\Big[-\int_0^t g_{ij}D_j\hat{v}_i(x,\tau)d\tau\Big]\qquad(D_j=\partial/\partial x_j). \tag{1.7}$$

(It is noted that the initial-boundary conditions for $\hat{v}(x,t)$ etc. are the same as those for $v(x,t)$ etc.) Extending $\theta_1\in H^{2+\alpha}(\Gamma_T)$ to $\theta_1^*\in H^{2+\alpha}(\bar{Q}_T)$ and setting $w_i(x,t)=\hat{v}_i(x,t)-v_{0i}(x)$ $(i=1,2,3)$, $w_4(x,t)=\hat{\theta}(x,t)-\theta_1^*(x,t)+\theta_1^*(x,0)-\theta_0(x)$, from (1.2), (1.3′), (1.4) and the above arguments we derive

$$\begin{cases}D_tw=\mathfrak{A}(x,t,w)D_x^2w+\mathfrak{B}\Big(x,t,w,D_xw,\int_0^tD_x^2w\,d\tau\Big),\\ w(x,0)=0,\qquad w|_{\Gamma_T}=0.\end{cases}\tag{1.8}$$

Secondly we make the following linear problem correspond with (1.8):

$$\begin{cases}D_t\tilde{w}=\mathfrak{A}(x,t,w)D_x^2\tilde{w}+\mathfrak{B}\Big(x,t,w,D_xw,\int_0^tD_x^2w\,d\tau\Big),\\ \tilde{w}(x,0)=0,\qquad \tilde{w}|_{\Gamma_T}=0,\end{cases}\tag{1.9}$$

where $\mathfrak{A}, \mathfrak{B} \in H^{\alpha}(\bar{Q}_T)$ and

$$w \in \mathfrak{S}_T \equiv \Big\{ w \in H^{2+\alpha}(\bar{Q}_T) \,|\, w|_{\Gamma_T}=0,\ \langle w \rangle_T^{(2,\alpha)} \equiv \sum_{2r+|s|=0}^{2} |D_t^r D_x^s w|_T^{(0)} + \sum_{|s|=1} |D_x^s w|_{t,T}^{(\alpha/2)} < M_1 \Big\} \qquad (\forall M_1 \in (0, \bar{\theta}_0)).$$

Then there exists $T_1 \in (0, T]$ such that the system (1.9) is uniformly parabolic in Petrowsky's sense in $\bar{Q}_{T_1}$.

2.1. The Green matrix and its estimates. First of all we consider the problem:

$$\text{(2.1)} \qquad \begin{cases} D_t W = \mathfrak{A}(x, t, w) D_x^2 W & \text{in } Q_{\tau,T} = \Omega \times (\tau, T], \\ W|_{t=\tau} = 0, \quad W|_{\Gamma_{\tau,T}} = Z|_{\Gamma_{\tau,T}} & (\Gamma_{\tau,T} = \Gamma \times [\tau, T]), \end{cases}$$

where Z is a fundamental solution for the extended system of (1.9) in R^3. By a local coordinate $\{\bar{x}\}$, (2.1) is transformed into a system of the same type in a half space $R^3_+ = \{\bar{x}_3 \geqq 0\}$. After lengthy calculations, we can check that Lopatinsky's condition for the transformed system is satisfied. Hence the solution G_0 of (2.1) can be constructed and then the Green matrix G is defined by $G = Z - G_0$, which is evaluated as follows, e.g.,

$$\text{(2.2)} \quad |D_t^r D_x^s G(x, t; \xi, \tau; w)| \leqq C_1^{(r,|s|)} (t-\tau)^{-(3+2r+|s|)/2} \cdot \exp\left[-d_1 \frac{|x-\xi|^2}{t-\tau} \right] \qquad (2r + |s| \leqq 2).$$

2.2. Estimates of the bounded solution of a linear problem.

$$\text{(2.3)} \quad \tilde{w}(x, t) = \int_0^t d\tau \int_\Omega G(x, t; \xi, \tau; w) \mathfrak{B}\Big(\xi, \tau, w, D_\xi w, \int_0^\tau D_\xi^2 w \, d\tau_0\Big) d\xi$$

is obviously a solution of (1.9) and we have, e.g., for $|s| = 1, 2$

$$\text{(2.4)} \qquad |D_x^s \tilde{w}(x, t) - D_x^s \tilde{w}(x, t')| \leqq C_2^{(|s|)} (t-t')^{-(2-|s|+\alpha)/2} \, \|\mathfrak{B}\|_{T_1}^{(\alpha)}$$

($\|\cdot\|_T^{(\alpha)} = |\cdot|_T^{(0)} + |\cdot|_T^{(\alpha)}$). $C_2^{(|s|)}$ are positive functions continuous in $\langle w \rangle_{T_1}^{(2,\alpha)}$, T_1 and initial data and monotonically increasing in each argument. From the estimate of $\|\mathfrak{B}\|_{T_1}^{(\alpha)}$ it follows that there exist $T_2 \in (0, T_1]$ and $M_2 (>0)$ such that

$$\text{(2.5)} \qquad |D_x^2 \tilde{w}|_{T_2}^{(\alpha)} \leqq M_2, \quad \text{or,} \quad \tilde{w} \in \mathfrak{S}_{T_2}^0 \equiv \{ w \in \mathfrak{S}_{T_2} \,|\, |D_x^2 w|_{T_2}^{(\alpha)} \leqq M_2 \}.$$

3. The existence and uniqueness of a bounded solution of (1.8). Let us define a sequence $\{w_n\}$ such that $w_0(x, t) = 0$ ($\in \mathfrak{S}_T^0$) and $w_n(x, t)$ be a solution of (1.9) with $w = w_{n-1} \in \mathfrak{S}_T^0$. Then the above arguments imply $w_n \in \mathfrak{S}_T^0$. According to the estimates of $\|\mathfrak{A}(x, t, w_{n-1}) - \mathfrak{A}(x, t, w_{n-2})\|_T^{(\alpha)}$ and $\|\mathfrak{B}(x, t, w_{n-1}) - \mathfrak{B}(x, t, w_{n-2})\|_T^{(\alpha)}$ and the estimates in the previous section we have, on the basis of the expression that $w_n - w_{n-1}$ satisfies,

$$\text{(3.1)} \quad \langle\!\langle w_n - w_{n-1} \rangle\!\rangle_T^{(2,\alpha)} \leqq C_3(T, \langle\!\langle w_{n-1} \rangle\!\rangle_T^{(2,\alpha)} + \langle\!\langle w_{n-2} \rangle\!\rangle_T^{(2,\alpha)}) \langle\!\langle w_{n-1} - w_{n-2} \rangle\!\rangle_T^{(2,\alpha)}$$

($\langle\!\langle \cdot \rangle\!\rangle_T^{(2,\alpha)} = \langle \cdot \rangle_T^{(2,\alpha)} + |D_x^2 \cdot|_T^{(\alpha)}$). Since $C_3 \downarrow 0$ as $T \downarrow 0$, we can take $T' \in (0, T]$ such that $C_3(T', 2M_1 + 2M_2) < 1$. Thus w_n uniformly converges to $w \in H^{2+\alpha}(\bar{Q}_{T'})$ as $n \to \infty$. From (1.7) it follows that $\hat{\rho}(x, t; w_n + v_0)$ converges to $\hat{\rho}(x, t; w + v_0) \in B^{1+\alpha}(\bar{Q}_{T'})$. The proof of uniqueness is given by the same estimate as (3.1). Now we have

Theorem 1. *For some* $T' \in (0, T]$ $\exists_1 (w, \rho) \in H^{2+\alpha}(\bar{Q}_{T'}) \times B^{1+\alpha}(\bar{Q}_{T'})$ *such that* $(w, \hat{\rho})$ *satisfis* (1.8).

Theorem 2. *For some* $T' \in (0, T]$, $\exists_1 (v, \theta, \rho) \in H^{2+\alpha}(\bar{Q}_{T'}) \times H^{2+\alpha}(\bar{Q}_{T'}) \times B^{1+\alpha}(\bar{Q}_{T'})$ *such that* (v, θ, ρ) *satisfies* (1.1), (1.2), (1.3′) *and* (1.5).

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