129. On the Sum of the Möbius Function in a Short Segment

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1. Let $\mu(n)$ be the Möbius function and let

$$M(x)=\sum_{n\leqq x}\mu(n).$$

Then by the familiar device*)

$$\zeta(s)^{-1}=\zeta(s)^{-1}(1-\zeta(s)H(s))^2+2H(s)-\zeta(s)H(s)^2,$$

where

$$H(s)=\sum_{n\leqq y}\mu(n)n^{-s}$$

with certain y, we can prove that there is an absolute constant $\vartheta, 0<\vartheta<1$, such that

(1) $$M(x+h)-M(x)=o(h) \qquad (\text{as } x\to\infty)$$

uniformly for $h, x\geqq h\geqq x^{\vartheta}$. But it seems that by this method it is very difficult to get a result which corresponds to Huxley's estimate [3] of the discrepancy between consecutive primes.

In this note we indicate very briefly that there is an alternative way to prove such a result. Our result is as follows:

Theorem. (1) *is true, whenever* $\vartheta>7/12$.

2. Now we show only the main steps of our argument.

We have

(2) $$M(x+h)-M(x)=\frac{1}{2\pi i}\int_l \zeta(s)^{-1}((x+h)^s-x^s)s^{-1}ds+O(x/T),$$

where l is the straight line connecting the points $1-\delta+iT$ and $1-\delta-iT$, T being sufficiently large and $\delta=(\log T)^{-2/3-\varepsilon}$ with arbitrary small positive constant ε. Here we have used Vinogradov's estimate of the zero-free region of $\zeta(s)$. Let

$$\mathcal{D}=\bigcup_{j=0}^{J}\bigcup_{k=-K}^{K}\Delta(j,k),$$

where $J=[(1/2-\delta)\log T]$, $K=[T(\log T)^{-1}]$ and

$$\Delta(j,k)=\{s=\sigma+it\,;\,\sigma_j\leqq\sigma<\sigma_{j+1},\ k(\log T)\leqq t<(k+1)\log T\},$$

σ_j being $1/2+j(\log T)^{-1}$. We divide $\Delta(j,k)$ into two classes (W) and (Y) as follows: When $\sigma_j\leqq 1-\varepsilon$, then $\Delta(j,k)\in(W)$ if and only if $\Delta(j,k)$ contains at least one zero of $\zeta(s)$, and the remaining rectangles go into

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*) In recent literature this kind of modification has been attributed to Gallagher [1], but this seems originally due to Heilbronn [2].

(Y). On the other hand when $1-\varepsilon<\sigma_j\leqq 1-\delta$, then $\Delta(j,k)\in(W)$ if and only if there is at least one $s\in\Delta(j,k)$ such that

$$|\zeta(s)G_j(s)|<1/2,$$

where

$$G_j(s)=\sum_{n\leqq X_j}\mu(n)n^{-s},$$

$$X_j=\left\{(\log T)^5 \operatorname*{Max}_{\substack{\sigma\geqq 4\sigma_j-3\\ 1\leqq|t|\leqq T}}|\zeta(s)|\right\}^{1/(2(1-\sigma_j))},$$

and $\Delta(j,k)\in(Y)$ if and only if for all $s\in\Delta(j,k)$

(3) $$|\zeta(s)G_j(s)|\geqq 1/2.$$

Then by Huxley [3] we have

(4) $$\sharp\{k\,;\,\Delta(j,k)\in(W)\}\ll T^{12(1-\sigma_j)/5}(\log T)^9$$

if $\sigma_j\leqq 1-\varepsilon$. And by the argument of Montgomery [4; pp. 110–112] we have, if $1-\varepsilon<\sigma_j\leqq 1-\delta$,

(5) $$\sharp\{k\,;\,\Delta(j,k)\in(W)\}\ll X_j^{10(1-\sigma_j)/3}(\log T)^6 \ll T^{c(1-\sigma_j)^{3/2}}(\log T)^{16},$$

where we have used Richert's estimate [5].

Now let $j_k=\operatorname{Max}\{j\,;\,\Delta(j,k)\in(W)\}$, and let

$$\mathscr{D}'=\bigcup_{k=-K}^{K}\bigcup_{j\leqq j_k}\Delta(j,k).$$

Further let $\mathscr{D}_0=\mathscr{D}-\mathscr{D}'$. We write in $\mathscr{D}_0$ the line L which consists of vertical and horizontal segments: The horizontal segments keep the distances $\log\log T$ from $\mathscr{D}'$. And the vertical segments keep the distances ε^2 if $\sigma\leqq 1-\varepsilon$, and $(\log T)^{-1}$ if $1-\varepsilon<\sigma\leqq 1-\delta$. Then as in [6; pp. 282–283] we have, appealing to the Borel-Carathéodory and Hadamard's three circle theorems,

$$\zeta(s)^{-1}\ll\exp\left((\log T)^{1-\varepsilon^2}\right)$$

if $s\in L$ and $\sigma\leqq 1-\varepsilon+\varepsilon^2$. Also by (3) we have

$$\zeta(s)^{-1}\ll G_j(s)\ll X_j^{1-\sigma_j}\log T \ll\exp\left(c(1-\sigma_j)^{3/2}\log T\right)(\log T)^4$$

if $s\in L$ and $1-\varepsilon<\sigma_j\leqq\sigma<\sigma_{j+1}$. Thus we see that we have, for all $s\in L$,

$$\zeta(s)^{-1}\ll\exp\left(c\sqrt{\varepsilon}(1-\sigma)\log T\right)(\log T)^4.$$

Now, returning to (2) and observing (4), (5), we have

$$\begin{aligned}M(x+h)-M(x)&\ll h\int_L|\zeta(s)^{-1}|\,x^{\sigma-1}\,|ds|+O(x/T)\\&\ll hx^{\varepsilon^2}(\log T)^{13}\sum_{j=0}^{J_1}\exp\left((1-\sigma_j)((12/5+c\sqrt{\varepsilon})\log T-\log x)\right)\\&\quad+h(\log T)^{22}\sum_{j=J_1+1}^{J}\exp\left((1-\sigma_j)(2c\sqrt{\varepsilon}\log T-\log x)\right)\\&\quad+O(x/T),\end{aligned}$$

where $J_1=[(1/2-\varepsilon)\log T]$.

Finally setting $T=\exp\left((5/12)(1-c\sqrt{\varepsilon}/12)\log x\right)$ we end the proof.

Concluding remark. Similarly, but much easier than, we can prove, denoting by p_n the n-th prime,

$$p_{n+1}-p_n<p_n^{7/12}(\log p_n)^{25}$$

for sufficiently large n.

References

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