30. *Some Derived Rules of Intuitionistic Second Order Arithmetic*

By Susumu HAYASHI
Department of Mathematics, St. Paul's University, Tokyo

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L. E. J. Brouwer introduced some intuitionistic principles in his study of intuitionistic analysis. They cannot be proved in the intuitionistic second order arithmetic. Moreover some of them are incompatible with the classical mathematics when they are interpreted in the classical sense. But it has been shown, that the derived rules which correspond to some of Brouwer's principles are valid for various intuitionistic systems (e.g. [1], [2]).[1] The purpose of this note is to announce the result that the derived rules which correspond to Brouwer's principles are valid for the intuitionistic second order arithmetic. Details will be published elsewhere. The author would like to thank Professor Troelstra for his helpful suggestion.

Let $\boldsymbol{S}$ be the formal system of intuitionistic second order arithmetic with equality (i.e. ***HAS*** of [1]). In this system one can define the notions of functions and real numbers in the sense of Cauchy sequence of rational numbers. We use $f, g, h, \cdots$ as the variables for functions from natural numbers to natural numbers, and use $x, y, z, m, n, \cdots$ as the variables for natural numbers. $A, B, \cdots$ stand for formulae of $\boldsymbol{S}$. $\bar{f}(x)$, $(h|f)$, $f(g)$, $!\,f(g)$, $!\,(h|f)$, and $f \leq g$ will be used in the following sense:

$$\begin{cases} \bar{f}(0) =_{\mathrm{def}} \langle\ \rangle \\ \bar{f}(n+1) =_{\mathrm{def}} \langle f(0), \cdots, f(n) \rangle \end{cases}$$

$$(h|f)(x) \simeq y \equiv_{\mathrm{def}} h(\langle x \rangle * \bar{f}(\min_z [h(\langle x \rangle * \bar{f}(z)) > 0])) \dot{-} 1 = y$$

$$f(g) \simeq y \equiv_{\mathrm{def}} f(\bar{g}(\min_z [f(\bar{g}(z)) > 0])) - 1 = y$$

$$!\,f(g) \equiv_{\mathrm{def}} \exists y (f(g) \simeq x)$$

$$!\,(h|f) \equiv_{\mathrm{def}} \forall x \exists y (y \simeq (h|f)(x))$$

$$f \leq g \equiv_{\mathrm{def}} \forall x (f(x) \leq g(x)).$$

Theorem 1 (Continuity Rule). *If* $\boldsymbol{S} \vdash \forall f \exists g A(f, g)$, *then* $\boldsymbol{S} \vdash \exists h \{h$ *is primitive recursive* & $\forall f(!\,(h|f)$ & $A(f, (h|f))\}$.

Theorem 2 (Fan Rule). *If* $\boldsymbol{S} \vdash \forall f \exists n A(f, n)$, *then* $\boldsymbol{S} \vdash \exists f[f$ *is primitive recursive* & $\forall g\{!\,f(g) \& (\forall h \leq g \exists n \forall k (\bar{h}(f(g)) = \bar{k}(f(g)) \to A(k, n))\}]$.

Theorem 3. *Let* $\boldsymbol{R}$ *be the whole of real numbers.* (i) *If* $\boldsymbol{S} \vdash \{A$ *is*

1) Professor Troelstra informed the author that the Bar Induction Rule for $\boldsymbol{HA}^{\omega}$ has been obtained by H. Schwichtenberg in his unpublished inaugural dissertation (1973).

a function from the interval $[0,1]$ *to* $\boldsymbol{R}$}, *then* $\boldsymbol{S}\vdash\{A$ *is uniformly continuous on* $[0,1]\}$. (ii) *If* $\boldsymbol{S}\vdash\{A$ *is a function from* $\boldsymbol{R}$ *to* $\boldsymbol{R}\}$, *then* $\boldsymbol{S}\vdash\{A$ *is continuous on* $\boldsymbol{R}\}$.

Theorem 4 (Transfinite Induction Rule). *Let* ρ *be a formula of* $\boldsymbol{S}$, *and* $WF(\rho)$ *and* $I(\rho)$ *be the formulae*

$$\forall f \exists n \neg (f(n) \rho f(n+1))$$

and

$$\forall Q^{(0)}[\forall x\{\forall y(x\rho y \to Qy) \to Qx\} \to \forall x Qx],$$

respectively. Let $\boldsymbol{S}^c$ *be the system obtained from* $\boldsymbol{S}$ *by adjoining the axiom* $\forall X^1(X \vee \neg X)$. (i) *If* $\boldsymbol{S}\vdash WF(\rho)$, *then* $\boldsymbol{S}\vdash I(\rho)$. (ii) *If* $\boldsymbol{S}\vdash\forall xy(x\rho y \vee \neg x\rho y)$ *and* $\boldsymbol{S}^c\vdash WF(\rho)$, *then* $\boldsymbol{S}\vdash I(\rho)$.

Theorem 5 (Bar Induction Rule). *Let* A *be a formula of* $\boldsymbol{S}$, *and* $H1$, $H2$, *and* $H3$ *be the formulae*

$$\forall fn[A(f,n) \to A(f,n+1)]$$
$$\forall fn[A(f,n) \to Q(\bar{f}(n))]$$

and

$$\forall x[(\forall y Q(x*\langle y\rangle)) \to Q(x)],$$

respectively. If $\boldsymbol{S}\vdash\forall f\exists n A(f,n)$, *then* $\boldsymbol{S}\vdash\forall Q^{(0)}[\{H1 \,\&\, H2 \,\&\, H3\} \to Q(\langle\ \rangle)]$.

Theorem 6 (Markov's Rule of type 1). *If* $\boldsymbol{S}\vdash\forall f g(A(f,g) \vee \neg A(f,g))$ *and* $\boldsymbol{S}^c\vdash\forall f\exists g A(f,g)$, *then* $\boldsymbol{S}\vdash\forall f\exists g A(f,g)$, *where* $\boldsymbol{S}^c$ *has the same meaning as in Theorem* 4.

Remark 1. Let $\boldsymbol{S}'$ be the arithmetic whose logical base is the intuitionistic simple type theory with the axioms of extensionality. When $\boldsymbol{S}$ is replaced by $\boldsymbol{S}'$ in the above theorems all of those also hold true. In $\boldsymbol{S}'$ one can define the notions of functions of any finite types, thence he can define rules for higher types, e.g. the Bar Induction Rules of type $\sigma \neq 0$ and the Transfinite Induction Rules of type $\sigma \neq 0$ (cf. [3], [4]). Let Ω be the set defined by

$$0 \in \Omega; \quad \text{if } \sigma \in \Omega, \text{ then } (0 \to \sigma) \in \Omega.$$

If $\sigma \in \Omega$ and $\sigma \neq 0$, functions of type σ may be identified with functions of type 1, e.g. let F be a function of type $0 \to (0 \to 0)$, then F may be identified with the function $\lambda x^0(F(j_1(x))(j_2(x)))$, where $\langle j_1(x), j_2(x)\rangle = x$. Thence for types which belong to Ω we can easily prove the Transfinite Induction Rule and the Bar Induction Rule. The Continuity Rule of type σ, i.e. if $\boldsymbol{S}'\vdash\forall F^{0\to\sigma}\exists n A(F,n)$, then $\boldsymbol{S}'\vdash\forall F^{0\to\sigma}\exists n\exists x\forall G^{0\to\sigma}[\forall y<n(F(y)=G(y)) \to A(G,x)]$, is valid if and only if $\sigma \in \Omega$.[2] It seems that for types which do not belong to Ω the Bar Induction Rule and the Transfinite Induction Rule for $\boldsymbol{S}'$ are not valid (cf. [4]), however, the author has not found a solution.

2) It is easy to see, if one can find a counterexample for $\sigma=2$, then he can also find a counterexample for every type $\sigma \notin \Omega$. For $\sigma=2$, $\lambda y^{0\to 2}((y0)(\lambda x((yx)\lambda z^0 0)))$ is a counterexample, since this function exists probably in $\boldsymbol{S}'$, but in the corresponding classical system it can be proved that this function is not continuous.

Remark 2. The results of the present note are proved by formalizations of analyses of infinitary normal forms of proofs of $\boldsymbol{S}$. To prove the normalization theorem, for each proofs of $\boldsymbol{S}$ we define a tree which is labelled by proofs of $\boldsymbol{S}$ and is called the *normalization tree* of the proof. It is a generalization of Martin-Löf's normalization (see [5]).

References

[1] A. S. Troelstra: Metamathematical investigation of intuitionistic arithmetic and analysis. Springer Lecture Notes, **344** (1973).

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[4] W. A. Howard and G. Kreisel: Transfinite induction and bar induction of type zero and one, and the role of continuity in intuitionistic analysis. Journal of Symbolic Logic, **31**, 325–358 (1966).

[5] P. Martin-Löf: Hauptsatz for the intuitionistic theory of iterated inductive defenitions. Proceedings of the Second Scandinavian Logic Symposium, North-Holland, 179–216 (1971).