# 15. On Commutators of Equivariant Diffeomorphisms

By Kōjun ABE*) and Kazuhiko FUKUI**)

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**1. Introduction and statement of the result.** J. N. Mather [3] and W. Thurston [5] have shown that certain groups of diffeomorphisms of smooth manifolds are perfect, i.e., equal to their own commutator subgroups. In this note, we shall prove that certain groups of equivariant diffeomorphisms of principal $G$-manifolds are perfect, where $G$ is a compact Lie group. We note that the case $G=T^q$ is discussed by A. Banyaga [2] precedently.

Let $M$ be a smooth manifold without boundary on which a compact Lie group $G$ acts smoothly and freely. Let $\mathrm{Diff}_G^r(M)_0$ be the group of equivariant $C^r$ diffeomorphisms of $M$ which are $G$-isotopic to the identity through compactly supported equivariant $C^r$ isotopies.

**Theorem.** *If* $1\leq r\leq\infty$, $r\neq m-q+1$ *and* $m-q\geq 1$, *then* $\mathrm{Diff}_G^r(M)_0$ *is perfect, where* $m=\dim M$ *and* $q=\dim G$.

**2. Fragmentation lemma.** Let $K$ be a compact subset of $M$ and let $\mathrm{Diff}_{G,K}^r(M)_0$ be a group of equivariant $C^r$ diffeomorphisms of $M$ which are $G$-isotopic to the identity through an equivariant $C^r$ isotopies whose supports are contained in $K$, with $C^r$ topology.

**Lemma 1** (cf. J. Palis and S. Smale [4] Lemma 3.1). *Let* $\{V_i; 1\leq i<n\}$ *be a G-invariant finite open covering of M and let N be an open neighborhood of the identity in* $\mathrm{Diff}_{G,K}^r(M)_0$. *Then there exists an open neighborhood* $N_0\subset N$ *of the identity with the following properties: For any* $f\in N_0$, *there exist* $f_i\in N$, $1\leq i\leq n$, *such that*

a) $f_i$ *is G-isotopic to the identity through an equivariant* $C^r$ *isotopy whose support is contained in* $V_i\cap K$, *and*

b) $f=f_n\cdot f_{n-1}\cdots f_1$.

Since $\mathrm{Diff}_G^r(M)_0=\bigcup_K \mathrm{Diff}_{G,K}^r(M)_0$, Lemma 1 reduces the proof of Theorem to the special case when $M=R^{m-q}\times G$, where $R^{m-q}$ is equipped with the trivial $G$-action.

**3. Some lemmas.** Let $U=R^{m-q}$ and let $\pi: U\times G\to U$ be a natural projection. Let $\mathrm{Diff}^r(U)_0$ be the group of $C^r$ diffeomorphisms of $U$ which are isotopic to the identity through compactly supported $C^r$ isotopies. Let $P: \mathrm{Diff}_G^r(U\times G)_0\to\mathrm{Diff}^r(U)_0$ be a homomorphism given by $P(h)(x)=\pi(h(x,1))$, for $h\in\mathrm{Diff}_G^r(U\times G)_0$ and $x\in U$. Let $G_0$ be the

*) Department of Mathematics, Shinshu University.
**) Department of Mathematics, Kyoto Sangyo University.

identity component of the Lie group $G$. Let $e$ be a constant mapping defined by $e(x)=1$ for $x \in U$. For a map $f: U \to G_0$, $\operatorname{supp}(f)$ denotes the closure of $f^{-1}(G_0-\{1\})$. Let $C^r(U, G_0)_0$ denote the set of $C^r$ maps $f: U \to G_0$ which are $C^r$ homotopic to $e$ through compactly supported $C^r$ homotopies, with $C^r$ topology.

**Lemma 2.** *Let $L: \operatorname{Ker} P \to C^r(U, G_0)_0$ be a homomorphism defined by the following equality: $h(x, 1)=(x, L(h)(x))$, for $h \in \operatorname{Ker} P$ and $x \in U$. Then $L$ is an isomorphism.*

The following lemma plays a key role in the proof of Theorem.

**Lemma 3.** *For $\delta>0$, let $B_\delta$ be the ball in $R^n$ of radius $\delta$, centered at 0. Let $u: R^n \to R$, $n \geq 1$, be a $C^r$ function supported in $B_\delta$ which is $C^1$-close to the zero map. Then there exist a $C^\infty$ function $v: R^n \to R$ supported in $B_{\delta'}$ $(\delta'=2\sqrt{3}\,\delta)$, $|v(x)| \leq 3\delta$ for any $x \in U$, and a $C^r$ diffeomorphism $\phi: R^n \to R^n$ which is isotopic to the identity through a $C^r$ isotopy supported in $B_\delta$, such that $u=v \circ \phi - v$.*

**Proof.** We define a $C^r$ diffeomorphism $\phi: R^n \to R^n$ by $\phi(x_1, \cdots, x_n) =(x_1+u(x), x_2, \cdots, x_n)$ and the $C^r$ isotopy $\phi_t$, $0 \leq t \leq 1$, by $\phi_t(x_1, \cdots, x_n) =(x_1+tu(x), x_2, \cdots, x_n)$. Then $\phi_t$ is a $C^r$ isotopy supported in $B_\delta$ with $\phi_0=1_{R^n}$ and $\phi_1=\phi$. Let $\xi: R \to R$ be a $C^\infty$ function such that $\xi(x)=x$ if $|x| \leq 2\delta$, $|\xi(x)| \leq 3\delta$ if $2\delta \leq |x| \leq 3\delta$ and $\xi(x)=0$ if $|x| \geq 3\delta$. Let $\mu: R^{n-1} \to R$ be a $C^\infty$ function such that $0 \leq \mu(x) \leq 1$ for $x \in R^{n-1}$,

$$\mu(x_1, \cdots, x_{n-1})=1 \qquad \text{if } x_1^2+\cdots+x_{n-1}^2 \leq \delta^2,$$

and

$$\mu(x_1, \cdots, x_{n-1})=0 \qquad \text{if } x_1^2+\cdots+x_{n-1}^2 \geq 3\delta^2.$$

Let $v: R^n \to R$ be a $C^\infty$ function supported in $B_\delta$, defined by $v(x_1, \cdots, x_n) =\xi(x_1) \cdot \mu(x_2, \cdots, x_n)$. Then $u=v \circ \phi - v$.

**Remark.** In [1], A. Banyaga claimed that the above lemma holds when $u: R^n \to R^m$, $n, m \geq 1$, is a $C^r$ mapping supported in $B_\delta$. But his proof seems to be incorrect.

Let $L(G_0)$ be the Lie algebra associated to the Lie group $G_0$ and let $\{X_1, \cdots, X_q\}$ be a basis of $L(G_0)$. Let $\Phi: L(G_0) \to G_0$ be a mapping defined by $\Phi(a_1X_1+\cdots+a_qX_q)=(\exp a_1X_1)\cdots(\exp a_qX_q)$. Then there exist an open ball $V$ in $L(G_0)$ of radius $\varepsilon$, centered at 0 and a neighborhood $W$ of 1 in $G_0$ such that $\Phi$ is a diffeomorphism of $V$ onto $W$.

**Lemma 4.** *Let $f: U \to W$ be a $C^r$ mapping which is $C^1$-close to $e$, and the support of $f$ is contained in a ball of radius $\delta$ $(3\delta<\varepsilon)$ of $U$. Then there exist $f_i \in C^r(U, G_0)_0$ and $\phi_i \in \operatorname{Diff}^r(U)_0$, $i=1, \cdots, q$, such that $f=(f_1^{-1} \cdot (f_1 \circ \phi_1)) \cdots (f_q^{-1} \cdot (f_q \circ \phi_q))$.*

**Proof.** Let $\tilde{f}=\Phi^{-1} \circ f$. Then there exist $C^r$ functions $a_i: U \to R$, $i=1, \cdots, q$, such that $\tilde{f}(x)=a_1(x)X_1+\cdots+a_q(x)X_q$, for $x \in U$. By Lemma 3 there exist $C^r$ functions $v_i: U \to R$ with compact supports such that $|v_i(x)| \leq 3\delta$, for $x \in U$, and $\phi_i \in \operatorname{Diff}^r(U)_0$, $i=1, \cdots, q$, such that

$a_i=v_i\circ\phi_i-v_i$. Let $f_i: U\to W$ be a $C^r$ mapping defined by $f_i(x)=\exp(v_i(x)X_i)$ for $x\in U$. Then $f_i\in C^r(U, G_0)_0$ and $f=(f_1^{-1}\cdot(f_1\circ\phi_1))\cdots(f_q^{-1}\cdot(f_q\circ\phi_q))$.

For any $\phi\in \mathrm{Diff}^r(U)_0$, define $\tilde{\phi}\in \mathrm{Diff}^r_G(U\times G)_0$ by $\tilde{\phi}(x,g)=(\phi(x),g)$ for $x\in U$ and $g\in G$.

**Lemma 5.** *Let $h\in \operatorname{Ker} P$ and $f=L(h)$. For any $\phi\in\mathrm{Diff}^r(U)_0$, we have $L(h^{-1}\circ\tilde{\phi}^{-1}\circ h\circ\tilde{\phi})=f^{-1}\cdot(f\circ\phi)$.*

**4. Proof of Theorem.** By the same way as in Lemma 1, any element $f\in C^r(U, G_0)_0$ can be expressed as follows:

a) $f=f_s\cdot f_{s-1}\cdots f_1$, where $f_i: U\to W$, $1\leq i\leq s$, are $C^r$ mappings,

b) the support of $f_i$ is contained in a ball of $U$ of radius $\delta$, centered at $x_i$, for $x_i\in U$,

c) $f_i$ is $C^1$-close to $e$ as in Lemma 4.

Combining Lemmas 4 and 5, we have

**Proposition 6.** $\operatorname{Ker} P=[\operatorname{Ker} P, \mathrm{Diff}^r_G(U\times G)_0]$.

Since $1\to\operatorname{Ker} P\to\mathrm{Diff}^r_G(U\times G)_0\to\mathrm{Diff}^r(U)_0\to 1$ is exact, we have the following exact sequence:

$$\operatorname{Ker} P/[\operatorname{Ker} P, \mathrm{Diff}^r_G(U\times G)_0]\to H_1(\mathrm{Diff}^r_G(U\times G)_0)\to H_1(\mathrm{Diff}^r(U)_0)\to 0.$$

By the results of J. Mather [3] and W. Thurston [5], we have $H_1(\mathrm{Diff}^r(U)_0)=0$. Therefore, by Proposition 6, $\mathrm{Diff}^r_G(U\times G)_0$ is perfect. By the argument at the end of § 2, this completes the proof of Theorem.

**Corollary.** *Let $M$ be an $m$-dimensional smooth $G$-manifold without boundary with one orbit type. If $1\leq r\leq\infty$, $r\neq\dim M/G+1$ and $\dim M/G\geq 1$, then $\mathrm{Difi}^r_G(M)_0$ is perfect.*

**Proof.** Let $H$ be an isotopy subgroup of a point of the $G$-manifold $M$ and let $N(H)$ be the normalizer of $H$ in $G$. Let $M^H=\{x\in M; h\cdot x=x$ for $h\in H\}$. Then $\mathrm{Diff}^r_G(M)_0$ is isomorphic to $\mathrm{Diff}^r_{N(H)/H}(M^H)_0$ as a group. Since $M^H$ is a free $N(H)/H$-manifold, Corollary follows from our Theorem.

## References

[1] A. Banyaga: On the structure of the group of equivariant diffeomorphisms. Topology, **16**, 279–283 (1977).

[2] ——: Sur les groupe des automorphismes d'un $T^n$-fibré principal. C. R. Acad. Sc. Paris, **284**, Sér. A, 619–622 (1977).

[3] J. N. Mather: Commutators of diffeomorphisms I and II. Comment. Math. Helv., **49**, 512–528 (1974); **50**, 33–40 (1975).

[4] J. Palis and S. Smale: Structural stability theorems. Global Analysis (Symp. Pure. Math. XIV) Amer. Math. Soc., Providence R. I., 223–231 (1970).

[5] W. Thurston: Foliations and group of diffeomorphisms. Bull. Amer. Math. Soc., **80**, 304–307 (1974).