12. Scattering Solutions Decay at least Logarithmically

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1. Introduction. In this note we study the asymptotic behavior as $t\to\infty$ of scattering solutions of the equation

$$(1.1)\qquad \begin{cases} \dfrac{1}{i}\dfrac{\partial u}{\partial t}=Hu=\Big(P(D)+\sum_{j=1}^{N} q_j(x)Q_j(D)\Big)u \\ u(x,0)=\varphi(x). \end{cases}$$

(u is called a scattering solution if φ is orthogonal to any eigenfunction of H.) Here we assume

(A.1) $P(\xi)$ is an elliptic polynomial of degree m with real coefficients, which is bounded from below;

(A.2) $Q_j(\xi)$ is a polynomial of degree less than $m-1$;

(A.3) $q_j(x)e^{2a|x|}\in L_\infty(\boldsymbol{R}^n)$ for some $a>0$;

(A.4) the operator $\sum_{j=1}^{N} q_j(x)Q_j(D)$ is formally self-adjoint.

Under the assumptions (A.1)–(A.4) (hereafter referred to as (A)), the operator H with domain $H^m(\boldsymbol{R}^n)$, the usual Sobolev space, is self-adjoint in $L_2(\boldsymbol{R}^n)$.

By virtue of the result due to Kuroda [1] it follows from (A) that a scattering solution of (1.1) decays locally. Our problem is to study at what rate the solution decays. In [2] we investigated the problem under the condition (A.3)′ milder than (A.3).

(A.3)′ $q_j(x)(1+|x|^2)^s\in L_\infty(\boldsymbol{R}^n)$ for some $s>1/2$.

But in order to give an answer we assumed the non-existence of "generalized eigenvalues" of H. In this note, applying the method of Vainberg [5], we remove such assumption under the condition (A.3). (See also [3], in which a rather abstract answer was given without such assumption under the condition (A.3)′.) Our result is a generalization of that of Rauch [4], which states that a scattering solution of Schrödinger's equation in $\boldsymbol{R}^3$ decays like $t^{-1/2}$. We shall show that a scattering solution of Schrödinger's equation in $\boldsymbol{R}^{2k}$ decays at least logarithmically.

2. Results. Let Λ be the set of all critical values of $P(\xi)$: $\Lambda=\{P(\xi)\,;\,\xi\in\boldsymbol{R}^n,\ \mathrm{grad}\,P(\xi)=0\}$. (We note that Λ is a finite set.) Let $\Pi_\pm=\{z\in\boldsymbol{C}\,;\,\pm\mathrm{Im}\,z>0\}$ and $D_\pm=\boldsymbol{C}\backslash\{\lambda+i\mu\,;\,\lambda\in\Lambda,\ \mp\mu\geqq 0\}$. Let $\mathcal{H}$ be the Hilbert space of all holomorphic functions on the tublar domain $\boldsymbol{R}^n\times iB_a\subset\boldsymbol{C}^n$ ($B_a=\{\eta\in\boldsymbol{R}^n\,;\,|\eta|<a\}$) with norm $\|f\|_{\mathcal{H}}=\|f(\xi+i\eta)\|_{L_2(\boldsymbol{R}^n\times B_a)}<\infty$. Let $X=\boldsymbol{B}(\mathcal{H},\mathcal{H}^*)$ be the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}^*$. To state our theorem we introduce the following condition (B), which is satisfied in many interested cases.

(B) Let $h_{\pm j}(z)$ $(j=0,1,\cdots,N, z\in\Pi_{\pm})$ be the multiplication operators from $\mathcal{H}$ to $\mathcal{H}^*$: $h_{\pm 0}(z)=(P(\xi)-z)^{-1}$, $h_{\pm j}(z)=Q_j(\xi)\,(P(\xi)-z)^{-1}$ $(j=1,\cdots,N)$. Then there exists for any $\lambda\in\Lambda$ a complex neighborhood U_λ of λ such that $h_{\pm j}(z)$ admits a holomorphic extension onto $U_\lambda\cap D_\pm$ and has the form

$$(2.1)\qquad h_{\pm j}(z)=\sum_{k=0}^{K}\left[\sum_{l=1}^{r-1}A_{\pm j}^{k,l}w^{-l/r}+f_{\pm j}^{k}(w^{1/r})\right](\log w)^k,\qquad w=z-\lambda,$$

where $A_{\pm j}^{k,l}\in X$, $f_{\pm j}^{k}(\zeta)$ is an X-valued holomorphic function in a neighborhood of zero, $A_{\pm j}^{k,l}$ $(j\neq 0)$ and $f_{\pm j}^{k}(0)$ $(k\neq 0, j\neq 0)$ are of finite rank, and $r>0$ and $K\geqq 0$ are integers independent of λ. (Since Λ is a finite set, we may assume without loss of generality that r and K are independent of λ.)

Theorem 2.1. *Let* (A) *and* (B) *be satisfied. Then there exist integers $\sigma\geqq 0$ and γ such that for any φ which is orthogonal to all eigenfunctions of H*

$$(2.2)\qquad \|e^{-a|x|}e^{itH}\varphi\|_{L_2(\mathbf{R}^n)}\leqq C\,|t|^{-\sigma/r}\log^{-\gamma}|t|\,\|e^{a|x|}\varphi\|_{L_2(\mathbf{R}^n)},\qquad |t|>2.$$

Here $\gamma>0$ when $\sigma=0$, and $\gamma=0$ when $K=0$.

Remark 2.2. The integers σ and γ can be determined as follows. Let $C_{\pm\lambda}^{ij}$ be the operator given in the equality (3.1) (see Lemma 3.2 in § 3). Let $I=\bigcup_{\lambda\in\Lambda}I_\lambda$ and $I_\lambda=\{(i,j)\,;\,i+j>0, C_{\pm\lambda}^{ij}\neq 0\}\backslash\{i/r$ is a positive integer and $p_\lambda i=j\}$. Then

$$(2.3)\qquad \sigma=\min\{i\,;\,(i,j)\in I \text{ for some } j\}.$$

Let δ be a number which is equal to 1 when σ/r is a positive integer and equal to 0 otherwise. Then

$$(2.4)\qquad \gamma=\min\{\gamma_\lambda\,;\,\lambda\in\Lambda\},\qquad \gamma_\lambda=\min\{j+\delta-p_\lambda\sigma\,;\,(\sigma,j)\in I_\lambda\}.$$

Example 2.3. Let $P(\xi)$ be a homogeneous elliptic polynomial of degree m, and let $q(x)$ be a real-valued function with $q(x)e^{2a|x|}\in L_\infty(\mathbf{R}^n)$. Let $H=P(D)+q(x)$. Then the estimate (2.2) holds. Moreover we have

(i) when n is odd, $r=m$, $\gamma=0$, and σ is a positive integer with $\sigma/m\notin\mathbf{N}$;

(ii) when n is even, $r=m/2$;

(iii) if $m<n$ and zero is not a "generalized eigenvalue" (cf. [2]), $\sigma/r=n/m$ and $\gamma=0$;

(iv) if $m=n=2$ and the operator H satisfies the condition of Theorem 7 in [5], $\sigma/r=1$ and $\gamma=2$.

Example 2.4. Let $P(\xi)=(|\xi'|^2-1)^2+|\xi''|^4$ $((\xi',\xi'')\in\mathbf{R}^{n'}\times\mathbf{R}^{n''}=\mathbf{R}^n$, $n''\geqq 2)$, and let $q(x)$ be a real-valued function with $(|D_nq(x)|+|q(x)|)\cdot e^{2a|x|}\in L_\infty(\mathbf{R}^n)$. Let $H=P(D)+D_nq(x)D_n$. Then the estimate (2.2) holds. Moreover we have

(i) when n'' is odd, $r=4$, $\gamma=0$, and σ is a positive integer with $\sigma/4\notin\mathbf{N}$;

(ii) when n'' is even, $r=2$;

(iii) if 0 and 1 are not "generalized eigenvalues", $\sigma/r=1/2+n''/4$, and $\gamma=0$.

3. Proof. For the proof of Theorem 2.1 we prepare four lemmas. We denote by $F_{\pm}(z)$ the $\boldsymbol{B}(L_2)$-valued function $e^{-a|x|}(H-z)^{-1}e^{-a|x|}$ defined in $\Pi_{\pm}$.

Lemma 3.1. *Let* (A) *be satisfied. Then $F_{\pm}(z)$ is extended meromorphically to a complex neighborhood of $\boldsymbol{R}\backslash\Lambda$. The poles in $\boldsymbol{R}\backslash\Lambda$ are simple and contained in $\sigma_p(H)$ the point spectrum of H.*

Proof. For the proof of the first half we have only to remark that the X-valued function $(P(\xi)-z)^{-1}$ admits a holomorphic extension onto a neighborhood of $\boldsymbol{R}\backslash\Lambda$ (use the equation (A.1) in [1]), and that $\mathcal{F}^{-1}\mathcal{H}=\boldsymbol{H}=\{f\,;\,\|f\|_{\boldsymbol{H}}=\|e^{a|x|}(1+|x|)^{-(n+1)/4}f(x)\|_{L_2(\boldsymbol{R}^n)}<\infty\}$ (cf. [3]). We leave the proof of the latter half to the reader (cf. [1] and [4]).

Lemma 3.2. *Let* (A) *and* (B) *be satisfied. Then there exists for any $\lambda\in\Lambda$ a complex neighborhood U_λ of λ such that $F_{\pm}(z)$ admits a holomorphic extension onto $U_\lambda\cap D_{\pm}$ and has the form*

$$F_{\pm}(z)=w^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}C_{\pm\lambda}^{ij}[w^{1/r}(\log w)^{p_\lambda}]^i(\log w)^{-j},\qquad w=z-\lambda, \tag{3.1}$$

where $C_{\pm\lambda}^{ij}\in\boldsymbol{B}(L_2)$, p_λ is an integer, and the double series converges in the norm uniformly on $U_\lambda\cap D_{\pm}$. Moreover we have that $p_\lambda=0$ and $C_{\pm\lambda}^{ij}=0$ $(j\neq 0)$ when $K=0$.

Proof. We get the result along the line given in Lemma 1 in [5].

Lemma 3.3. *Let* (A) *be satisfied. Then there exist $b>0$ and $c>0$ such that $F_{\pm}(z)$ is holomorphic in $\{z\in\boldsymbol{C}\,;\,|\mathrm{Im}\,z|<b\,|\mathrm{Re}\,z|^{1-1/m}-c\}$, in which*

$$\|F_{\pm}^{(k)}(z)\|_{\boldsymbol{B}(L_2)}\leqq C_k\,|\mathrm{Re}\,z|^{-(k+1)(1-1/m)},\qquad k=0,1,\cdots. \tag{3.2}$$

Proof. Since $|\mathrm{Im}\,z^{1/m}|<d$ in $\{|\mathrm{Im}\,z|<md\,|\mathrm{Re}\,z|^{1-1/m}\}$, we get the result using the variation of Lemma 5.1 in [2]. (We assumed only for this lemma that $\deg Q_j\leqq m-2$.)

Lemma 3.4. *Let $z\in\boldsymbol{C}$, $k\in\boldsymbol{Z}$, and $l=0,1,\cdots$. Then the following estimate holds for any $|t|>2$.*

$$\left|\frac{d^l}{dt^l}\left[\int_{-\infty}^{\infty}(x\pm i0)^z\log^k(x\pm i0)e^{itx}dx\right]\right| \leqq\begin{cases}C\,|t|^{-\mathrm{Re}\,z-1-l}\log^k|t|, & z\neq 0,1,\cdots\\ C\,|t|^{-\mathrm{Re}\,z-1-l}\log^{k-1}|t|, & z=0,1,\cdots\text{ and }k\neq 0.\end{cases} \tag{3.3}$$

Proof. We shall prove (3.3) only in the case $k=-j<0$. If $\mathrm{Re}\,z<-(j+1)$, then we get (3.3) using the facts:

(i) $\mathcal{F}^{-1}((x\pm i0)^z)(t)=e^{\pm i(\pi/2)z}\Gamma^{-1}(-z)t_{\mp}^{-z-1}$;

(ii) $(x\pm i0)^z\log^{-j}(x\pm i0)=\displaystyle\int_0^{\mp i\infty}\frac{w^{j-1}}{(j-1)!}(x\pm i0)^{z-w}dw$;

(iii) $$\left|\frac{d^l}{dy^l}[\Gamma^{-1}(iy-z)]\right| \leqq C_l e^{(\pi/2)|y|}\, |y|^{\mathrm{Re}\, z+1/2} \log^l |y|, \quad y \in \boldsymbol{R},\ |y|>2,$$
$$l=0,1,\cdots.$$

The estimate (3.3) for $\mathrm{Re}\, z \geqq -(j+1)$ is an easy corollary of the above case.

Now let us prove the theorem. We set

$$G_\pm(y)=F_\pm(y)-\sum_{\mu\in\sigma_p(H)\cup\Lambda}\{[(z-\mu)F_\pm(z)]|_{z=\mu}\}(y-\mu)^{-1},$$
$$M(y)=(2\pi i)^{-1}(G_+(y)-G_-(y)),\ E=e^{-a|x|}.$$

Let χ be a C_0^∞-function which is equal to one on $[-R,R]$ for some sufficiently large R. Let φ be orthogonal to any eigenfunction of H and $E^{-1}\varphi \in L_2(\boldsymbol{R}^n)$. Then we obtain the following equality along the line given in [2] and [4].

$$\begin{aligned} Ee^{itH}\varphi &= \int_{-\infty}^{\infty} e^{ity}\chi(y)M(y)E^{-1}\varphi dy \\ &\quad -t^{-2}\int_{-\infty}^{\infty} e^{ity}\frac{d^2}{dy^2}[(1-\chi(y))M(y)]E^{-1}\varphi dy. \end{aligned} \tag{3.4}$$

This combined with Lemma 3.4 proves the theorem.

References

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