22. Residues and Secondary Characteristic Classes for Projective Foliations

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The purpose of this note is to explain the relation between residues of projective vector fields and secondary characteristic classes of projective foliations. As a consequence, in particular, we obtain the following

Theorem 1. *Let $BP\Gamma_n$ denote the classifying space of codimension n projective foliations and $H^*(PWO_n)\to H^*(BP\Gamma_n;\boldsymbol{R})$ be the characteristic homomorphism (see § 2 for the definition). Then there exist natural epimorphisms*

$$H_{2m-1}(BP\Gamma_{2m-1};\boldsymbol{Z})\to\boldsymbol{R}^{d(m)}\to 0,$$
$$\pi_{2m-1}(BP\Gamma_{2m-1})\to\boldsymbol{R}^{d(m)}\to 0,$$

where $d(m)=\dim H^{2m-1}(PWO_{2m-1})$.

The basic technique of the proof is to observe the continuous variation of secondary characteristic classes on the family of codimension $(2m-1)$ projective foliations defined by affine vector fields on $\boldsymbol{R}^{2m}$ with a single nondegenerate zero at the origin. Details and related topics will be published elsewhere.

1. Residues. Let (M,∇) be a C^∞ manifold with torsion-free connection ∇ and X be a projective vector field on M, a C^∞ vector field which generates a one-parameter group of local projective transformations. Let p be a zero of X in M. The Lie derivative operator $\mathcal{L}_X$ with respect to X induces a linear endomorphism L_p of the tangent space of M at p. X is called nondegenerate if L_p is nonsingular at each zero p of X.

For a nondegenerate projective vector field X with isolated zeros, the residue of X is defined as follows. Suppose M is even-dimensional, say $2m$, and is oriented. Denote by $^*L_p$ the skew-symmetric part of the linear endomorphism L_p with respect to a Riemannian metric g which is fixed once for all. Then for each $\mathrm{Ad}\,(GL_{2m})$-invariant polynomial $\phi\in I(GL_{2m})$ and each zero p of X, the ϕ-residue of X at p is defined by

$$\mathrm{Res}_\phi\,(X,p)=(\mathrm{Pf}\,(^*L_p)/\det\,(L_p))\phi(L_p),$$

where $\mathrm{Pf}\,(^*L_p)$ denotes the Paffian of $^*L_p$, the "square root" of $\det\,(^*L_p)$ determined by the orientation of M. Note that if X is in particular

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a Killing vector field on the Riemannian manifold (M, g), then the residue $\mathrm{Res}_\phi(X, p)$ reduces to the one defined by Bott [1].

The Lie algebra $\mathfrak{sl}_{2m}$ has a canonical graded Lie algebra structure $\mathfrak{sl}_{2m}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1=\boldsymbol{R}^{2m}+\mathfrak{gl}_{2m-1}+(\boldsymbol{R}^{2m})^*$ which identifies $\mathfrak{gl}_{2m-1}$ with the subalgebra $\mathfrak{g}_0$ of $\mathfrak{sl}_{2m}$ (cf. [5]). Let $I(GL_{2m}, SL_{2m})$ denote the subalgebra of $I(GL_{2m})$ generated by the elements whose restrictions to the canonically imbedded subalgebra $\mathfrak{gl}_{2m-1}\subset\mathfrak{gl}_{2m}$ can be extended to $\mathrm{Ad}\,(SL_{2m})$-invariant polynomials on $\mathfrak{sl}_{2m}\supset\mathfrak{gl}_{2m-1}$.

With these understood, we obtain

Theorem 2 (Residue formula). *Let (M, ∇) be a compact oriented C^∞ manifold of dimension $2m$ with torsion-free connection ∇. Let X be a nondegenerate projective vectr field on M with isolated zeros. Let $\phi\in I(GL_{2m}, SL_{2m})$ be an invariant polynomial of degree m. Then for the characteristic number of M defined by ϕ, we have*

$$(1/2\pi)^m\int_M \phi(\Omega)=\sum_{p\in \mathrm{Zero}\,(X)}\mathrm{Res}_\phi(X, p)$$

where Ω denotes the curvature form of ∇.

The proof depends on the strong vanishing property of characteristic forms in Lemma 1 in § 2 and the localization is accomplished by Stokes theorem.

2. Characteristic classes. Let $\mathcal{F}$ be a codimension n projective foliation on a C^∞ manifold M. $\mathcal{F}$ is by definition a maximal family of C^∞ submersions $f_\alpha: U_\alpha\to(\boldsymbol{R}^n, \nabla_\alpha)$ of open sets U_α in M to a Euclidean n-space $\boldsymbol{R}^n$ with torsion-free connections ∇_α such that the family $\{U_\alpha\}_\alpha$ is an open covering of M and for each $x\in U_\alpha\cap U_\beta$ there exists a local projective diffeomorphism $\gamma^x_{\beta\alpha}$ of $(\boldsymbol{R}^n, \nabla_\alpha)$ into $(\boldsymbol{R}^n, \nabla_\beta)$ satisfying $f_\beta=\gamma^x_{\beta\alpha}\circ f_\alpha$ in some neighborhood of x. Denote by $BP\Gamma_n$ the classifying space of codimension n projective foliations and let $f_{\mathcal{F}}: M\to BP\Gamma_n$ classify $\mathcal{F}$.

For codimension n projective foliations we can define their characteristic classes in the following manner (cf. Bott-Haefliger [2], Kamber-Tondeur [3] and Morita [4]). Define $c_i\in I(GL_n)$ by

$$c_i(A)=\mathrm{Tr}\,\{(A-(\mathrm{Tr}\,A/(n+1))I_n)^i\}+(-\mathrm{Tr}\,A/(n+1))^i,$$

where I_n is the $n\times n$ identity matrix and $A\in\mathfrak{gl}_n$. Let PWO_n denote the graded differential complex

$$PWO_n=\boldsymbol{R}[c_2, c_3, c_4, \cdots, c_n]/\{\phi\,|\deg\phi>n\}\otimes E(h_3, h_5, h_7, \cdots, h_l)$$

with $\deg c_i=2i$, $\deg h_i=2i-1$, $d(c_i\otimes 1)=0$ and $d(1\otimes h_i)=c_i\otimes 1$, where $\boldsymbol{R}[c_2, c_3, c_4, \cdots, c_n]$ is the polynomial algebra over $\boldsymbol{R}$ on the variables $c_2, c_3, c_4, \cdots, c_n$ and $E(h_3, h_5, h_7, \cdots, h_l)$ is the exterior algebra on the indicated variables $h_3, h_5, h_7, \cdots, h_l$, l is the largest odd integer $\leqq n$. Then there exists a universal homomorphism

$$\lambda^*: H^*(PWO_n)\to H^*(BP\Gamma_n; \boldsymbol{R}).$$

The homomorphism $\lambda^*_{\mathcal{F}}=f^*_{\mathcal{F}}\circ\lambda^*: H^*(PWO_n)\to H^*(M; \boldsymbol{R})$ is defined on cochain level as follows. Let $\nu(\mathcal{F})$ denote the normal bundle of $\mathcal{F}$. $\nu(\mathcal{F})$ has a canonical basic connection ω^b, a connection given by glueing together by a partition of unity the connection $f_\alpha^*\omega_\alpha$ induced on each $\nu(\mathcal{F})|U_\alpha$ from the connection form ω_α on TR^n by submersion $f_\alpha: U \to (R^n, \nabla_\alpha)$. Let ω^r be a metric connection on $\nu(\mathcal{F})$ and for each $t \in [0,1]$ form the connection $\omega^t = t\omega^b + (1-t)\omega^r$ on $\nu(\mathcal{F})$ and denote its curvature form by Ω_t. Then a map

$$\lambda_{\mathcal{F}} : PWO_n \to A^*(M)$$

of PWO_n to the de Rham complex $A^*(M)$ of M is defined by

$$\lambda_{\mathcal{F}}(c_i) = c_i(\Omega^b),$$

$$\lambda_{\mathcal{F}}(h_i) = \Delta_{c_i}(\omega^b, \omega^r) = i\int_0^1 c_i(\omega^b - \omega^r, \underbrace{\Omega_t, \cdots, \Omega_t}_{i-1})dt.$$

It follows from the following lemma and the homotopy formula

$$d\Delta_{c_i}(\omega^b, \omega^r) = c_i(\Omega^b) - c_i(\Omega^r)$$

that $\lambda_{\mathcal{F}}$ is in fact a DGA-homomorphism and hence induces $\lambda_{\mathcal{F}}^* : H^*(PWO_n) \to H^*(M; R)$.

Lemma 1 (cf. [5]). *Let* $\phi \in R[c_2, c_3, c_4, \cdots, c_n]$. *Then*

$$\phi(\Omega^b) = 0 \qquad \text{if } \deg \phi > n.$$

3. Continuous variation. Let $(R^{2m}, \langle\ ,\ \rangle)$ be a Euclidean $2m$-space with the standard flat metric $\langle\ ,\ \rangle$. Let X be an affine vector field on R^{2m} with a single nondegenerate zero at the origin 0, i.e. in local coordinates $X = \sum a_{ij}x^i\partial/\partial x^j$, $(a_{ij}) \in GL_{2m}$. Then X defines a codimension $(2m-1)$ projective foliation $\mathcal{F}_X$ on $M = R^{2m} - \{0\}$, which has the homotopy type of $(2m-1)$-sphere S^{2m-1}. Let f_X denote the classifying map of $\mathcal{F}_X$. The family of these projective foliations then gives rise to an independent continuous variation of secondary characteristic classes in $H^{2m-1}(PWO_{2m-1})$, from which Theorem 1 follows easily.

In fact, recall the universal homomorphism λ^*: $H^*(PWO_{2m-1}) \to H^*(BP\Gamma_{2m-1}; R)$ and define

$$\Phi : H_{2m-1}(BP\Gamma_{2m-1}; Z) \to R^{d(m)},$$

$$\Psi : \pi_{2m-1}(BP\Gamma_{2m-1}) \to R^{d(m)},$$

respectively by

$$\Phi(\alpha) = (\lambda^*\phi_1(\alpha), \cdots, \lambda^*\phi_{d(m)}(\alpha)), \qquad \alpha \in H_{2m-1}(BP\Gamma_{2m-1}; Z),$$

$$\Psi([f]) = (f^*(\lambda^*\phi_1)[S], \cdots, f^*(\lambda^*\phi_{d(m)})[S]), \qquad [f] \in \pi_{2m-1}(BP\Gamma_{2m-1}).$$

Here $d(m) = \dim H^{2m-1}(PWO_{2m-1})$, $[S]$ denotes a generator in $H_{2m-1}(M; Z)$ determined by S^{2m-1} and $\phi_1, \cdots, \phi_{d(m)}$ is an additive basis of $H^{2m-1}(PWO_{2m-1})$, each of which has a form $c_J h_i = c_{j_1}\cdots c_{j_k}h_i$, where $2(j_1 + \cdots + j_k) + 2i - 1 = 2m-1$, $j_1 \leqq \cdots \leqq j_k$, and $i \leqq j_1$ if $k > 0$.

To see the surjectivity of Φ and Ψ, evaluate Φ on $\alpha = (f_X)_*[S]$ and Ψ on $[f_X] \in [S^{2m-1}, BP\Gamma_{m-1}]$ respectively for various affine vector fields X on M. The value is then given by the following

Lemma 2. *Let* $f_X, \lambda, c_J h_i$ *be as above. Then*

$$f_X^* \lambda^*(c_J h_I)[S] = \mathrm{Res}_{c_J c_I}(X, 0),$$

where on the right hand side each c_k *is regarded as an element in* $I(GL_{2m})$ *defined by* $c_k(B) = \mathrm{Tr}\,\{(B - (\mathrm{Tr}\,B/2m)I_{2m})^k\}$.

References

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