25. *Cauchy Problems for Fuchsian Hyperbolic Partial Differential Equations*

By Hidetoshi TAHARA
Department of Mathematics, Sophia University

(Communicated by Kôsaku YOSIDA, M. J. A., April 12, 1978)

In this note, we deal with the Cauchy problems for the Fuchsian hyperbolic partial differential equations in the sense of Tahara [8] with differentiable coefficients and establish the existence and uniqueness theorem. Our theorem contains the non-characteristic Cauchy problems for a certain class of weakly hyperbolic equations with variable multiplicity and with non-smooth characteristic roots.

Let $P(t, x, \partial_t, \partial_x)$ be a linear partial differential operator of order m whose coefficients are differentiable functions on $\Omega=[0, T]\times \boldsymbol{R}^n$ $(T>0)$ of the form

$$P(t, x, \partial_t, \partial_x)=t^k\partial_t^m+P_1(t, x, \partial_x)t^{k-1}\partial_t^{m-1}+\cdots+P_k(t, x, \partial_x)\partial_t^{m-k}+P_{k+1}(t, x, \partial_x)\partial_t^{m-k-1}+\cdots+P_m(t, x, \partial_x),$$

where $(t, x)=(t, x_1, \cdots, x_n)\in[0, T]\times\boldsymbol{R}^n$. We assume the following conditions on P.

(A-1) $0\leqq k\leqq m$,

(A-2) $\operatorname{ord} P_j(t, x, \partial_x)\leqq j$ for $1\leqq j\leqq m$,

(A-3) $\operatorname{ord} P_j(0, x, \partial_x)\leqq 0$ for $1\leqq j\leqq k$.

Then P is said to be *of Fuchsian type with weight* $m-k$ *with respect to* t. From condition (A-3), $P_j(0, x, \partial_x)$ is a function for $1\leqq j\leqq k$. We set $P_j(0, x, \partial_x)=a_j(x)$ $(1\leqq j\leqq k)$. Then the indicial equation associated with P is defined by

$$\begin{aligned}\mathcal{C}(\lambda, x)&=\lambda(\lambda-1)\cdots(\lambda-m+1)+a_1(x)\lambda(\lambda-1)\cdots(\lambda-m+2)\\&\quad+\cdots+a_k(x)\lambda(\lambda-1)\cdots(\lambda-m+k+1)\\&=\lambda(\lambda-1)\cdots(\lambda-m+k+1)(\lambda-\rho_1(x))\cdots(\lambda-\rho_k(x)).\end{aligned}$$

The roots, which we denote by $\lambda=0, 1, \cdots, m-k-1, \rho_1(x), \cdots, \rho_k(x)$, are called the characteristic exponents of P. They are functions of x.

(A-4) (Coefficients.) Set $P_j(t, x, \partial_x)=\sum_{|\alpha|\leqq j} a_{j,\alpha}(t, x)\partial_x^\alpha$. Then

$$a_{j,\alpha}(t, x)\in\mathcal{B}^\infty(\Omega)$$

holds, where $\mathcal{B}^\infty(\Omega)$ is the space of all C^∞ functions with bounded derivatives on Ω.

(A-5) (Hyperbolicity.) Let $\tau_1(t, x, \xi), \cdots, \tau_m(t, x, \xi)$ be the roots of the equation

$$\tau^m+\sum_{j=1}^k\Big(\sum_{|\alpha|=j} a_{j,\alpha}(t, x)\xi^\alpha\Big)\tau^{m-j}+\sum_{j=k+1}^m\Big(\sum_{|\alpha|=j} t^{j-k}a_{j,\alpha}(t, x)\xi^\alpha\Big)\tau^{m-j}=0.$$

Then $\tau_i(t, x, \xi)$ is a real valued function on $\Omega\times(\boldsymbol{R}^n\backslash 0)$ for $1\leqq i\leqq m$.

(A-6) (Factorization.) For any $a_{j,\alpha}(t,x)$ such that $|\alpha|=j$, we have

$$\frac{a_{j,\alpha}(t,x)}{t^j} \in \mathscr{B}^\infty(\Omega), \qquad \text{if } 1 \leqq j \leqq k \text{ and } |\alpha|=j,$$

$$\frac{a_{j,\alpha}(t,x)}{t^k} \in \mathscr{B}^\infty(\Omega), \qquad \text{if } k+1 \leqq j \leqq m \text{ and } |\alpha|=j.$$

(A-7) (Basic quadratic form.) There exist a quadratic form $S(t,\xi)$ of $\xi=(\xi_1,\cdots,\xi_n)$ with parameter t and a positive constant c such that

$$|\tau_i(t,x,\xi)-\tau_j(t,x,\xi)| \geqq ct(S(t,\xi))^{1/2}$$

holds for $i \neq j$, $(t,x,\xi) \in \Omega \times (\boldsymbol{R}^n \backslash 0)$.

Here we assume the following conditions on $S(t,\xi)$. We set $S(t,\xi) = \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j$, then

(i) $a_{ij}(t) \in C^1([0,T])$, $a_{ij}(t)=a_{ji}(t)$ and $a_{ij}(t)$ is a real valued function,

(ii) for any $t>0$, $S(t,\xi)$ is a positive definite quadratic form of ξ, and

(iii) $\max_{|\xi|=1} |\partial_t \log S(t,\xi)| = O\left(\frac{1}{t}\right)$ as $t \to +0$.

Moreover we assume the following estimates on P.

(A-8) (Estimates of principal part.) For any multi-index β, there exists a positive constant C_β such that (i) and (ii) are valid:

(i) If $1 \leqq j \leqq k$, then the following estimates hold on $\Omega \times (\boldsymbol{R}^n \backslash 0)$:

$$\left|\partial_x^\beta \sum_{|\alpha|=j} a_{j,\alpha}(t,x)\xi^\alpha\right| \leqq C_\beta t^j (S(t,\xi))^{j/2},$$

$$\left|\partial_t\partial_x^\beta \sum_{|\alpha|=j} a_{j,\alpha}(t,x)\xi^\alpha\right| \leqq C_\beta t^{j-1} (S(t,\xi))^{j/2}.$$

(ii) If $k+1 \leqq j \leqq m$, then the following estimates hold on $(0,T] \times \boldsymbol{R}^n \times (\boldsymbol{R}^n \backslash 0)$:

$$\left|\partial_x^\beta \sum_{|\alpha|=j} a_{j,\alpha}(t,x)\xi^\alpha\right| \leqq C_\beta t^k (S(t,\xi))^{j/2},$$

$$\left|\partial_t\partial_x^\beta \sum_{|\alpha|=j} a_{j,\alpha}(t,x)\xi^\alpha\right| \leqq C_\beta t^{k-1} (S(t,\xi))^{j/2}.$$

(A-9) (Estimates of lower order terms.) For any multi-index β, there exists a positive constant C_β such that (i) and (ii) are valid:

(i) If $1 \leqq j \leqq k$, then the following estimates hold on $\Omega \times (\boldsymbol{R}^n \backslash 0)$:

$$\left|\partial_x^\beta \sum_{|\alpha| \leqq j-1} a_{j,\alpha}(t,x)(\sqrt{-1}\xi)^\alpha\right| \leqq C_\beta (1+t^2 S(t,\xi))^{(j-1)/2}.$$

(ii) If $k+1 \leqq j \leqq m$, then the following estimates hold on $(0,T] \times \boldsymbol{R}^n \times (\boldsymbol{R}^n \backslash 0)$:

$$\left|\partial_x^\beta \sum_{|\alpha| \leqq j-1} a_{j,\alpha}(t,x)(\sqrt{-1}\xi)^\alpha\right| \leqq C_\beta t^{k-j} (1+t^2 S(t,\xi))^{(j-1)/2}.$$

(A-10) (Characteristic exponents.) There exists a positive constant c such that

$$|(\lambda-\rho_1(x))\cdots(\lambda-\rho_k(x))|\geqq\frac{c}{\lambda(\lambda-1)\cdots(\lambda-m+k+1)}$$

for any $x\in \boldsymbol{R}^n$ and $\lambda\in \boldsymbol{Z}$ such that $\lambda\geqq m-k$.

Under these assumptions (A-1)–(A-10), we have the next theorem.

Theorem. *For any functions* $u_0(x),\cdots,u_{m-k-1}(x)\in C^\infty(\boldsymbol{R}^n)$ *and* $f(t,x)\in C^\infty([0,T]\times\boldsymbol{R}^n)$, *there exists a unique solution* $u(t,x)\in C^\infty([0,T]\times\boldsymbol{R}^n)$ *such that*

$$\begin{cases}P(t,x,\partial_t,\partial_x)u(t,x)=f(t,x) & \text{on } \Omega,\\ \partial_t^i u(t,x)|_{t=0}=u_i(x) & \text{for } 0\leqq i\leqq m-k-1.\end{cases}$$

Moreover the solution has a finite propagation speed.

Remark 1. In the case of analytic coefficients, Cauchy problems for analytic solutions are solved in Hasegawa [3], Baouendi-Goulaouic [1], Tahara [8] under the assumptions (A-1), (A-2), (A-3) and (A-10). Tahara [8] also proved the existence and the uniqueness of hyperfunction solutions of the above Cauchy problems under the assumptions (A-1), (A-2), (A-3), (A-5), (A-6) and (A-10).

Remark 2. P is said to be *a Fuchsian hyperbolic operator with weight* $m-k$ *with respect to* t, if P satisfies the conditions (A-1), (A-2), (A-3), (A-5) and (A-6). See Tahara [7], [8].

Remark 3. If $k=0$, then (A-1), (A-2), (A-3), (A-6) and (A-10) are trivial and the above problems are nothing but the non-characteristic Cauchy problems for weakly hyperbolic equations. Even in the case $k=0$, our method, based on the study of solutions of partial differential equations with regular singularity, excels others in the following points.

(i) P may have a variable multiplicity.

(ii) We do not assume the smoothness of characteristic roots.

(iii) Mixed type operators such as Tricomi's operator can be treated in our framework.

These remarks will be illustrated in the following examples.

Examples. We will give here typical examples of our theory.

(1) (Tricomi's operator.) Let P be the operator

$$P(t,x,\partial_t,\partial_x)=\partial_t^2-t\partial_x^2+a(t,x)\partial_x+b(t,x)\partial_t+c(t,x),$$
$$a(t,x),\ b(t,x),\ c(t,x)\in\mathscr{B}^\infty(\Omega).$$

Then P satisfies our conditions with $m=2$, $k=0$, $S(t,\xi)=t\xi^2$.

(2) (Oleinik [6], Menikoff [5].) Let P be the operator

$$P(t,x,\partial_t,\partial_x)=\partial_t^2-t^{2l}\partial_x^2+t^{l-1}a(t,x)\partial_x+b(t,x)\partial_t+c(t,x),$$
$$a(t,x),\ b(t,x),\ c(t,x)\in\mathscr{B}^\infty(\Omega).$$

Then P satisfies our conditions with $m=2$, $k=0$, $S(t,\xi)=t^{2l}\xi^2$.

(3) (Euler-Poisson-Darboux equation, Delache-Leray [2].) Let P be the operator

$$P(t, x, \partial_t, \partial_x)=\partial_t^2-\partial_{x_1}^2-\cdots-\partial_{x_n}^2+\frac{\alpha}{t}\partial_t, \qquad \alpha \in \boldsymbol{C}.$$

Then tP satisfies our conditions with $m=2$, $k=1$, $S(t, \xi)=\xi_1^2+\cdots+\xi_n^2$.

(4) Let κ_i $(1 \leqq i \leqq n)$ be an integer $\geqq 0$. Let P be the operator

$$P(t, x, \partial_t, \partial_x)=\partial_t^m+\sum_{j=1}^m \Big(\sum_{|\alpha|=j} t^{\langle\kappa,\alpha\rangle} a_{j,\alpha}(t, x)\partial_x^\alpha\Big)\partial_t^{m-j}$$
$$+\sum_{j=1}^m \Big(\sum_{|\alpha| \leqq j-1} b_{j,\alpha}(t, x)\partial_x^\alpha\Big)\partial_t^{m-j},$$
$$\langle\kappa, \alpha\rangle=\kappa_1\alpha_1+\cdots+\kappa_n\alpha_n,$$
$$a_{j,\alpha}(t, x),\ b_{j,\alpha}(t, x) \in \mathcal{B}^\infty(\Omega).$$

We assume the following conditions (i), (ii) and (iii) on P:

(i) Let $\lambda_i(t, x, \xi)$ $(1 \leqq i \leqq m)$ be the root of the equation

$$\lambda^m+\sum_{j=1}^m \Big(\sum_{|\alpha|=j} a_{j,\alpha}(t, x)\xi^\alpha\Big)\lambda^{m-j}=0.$$

Then $\lambda_i(t, x, \xi)$ is a real valued function on $\Omega \times (\boldsymbol{R}^n \backslash 0)$ for $1 \leqq i \leqq m$.

(ii) There is a positive constant c such that

$$|\lambda_i(t, x, \xi)-\lambda_j(t, x, \xi)| \geqq c|\xi| \qquad (i \neq j).$$

(iii) For any $b_{j,\alpha}(t, x)$ $(|\alpha| \leqq j-1)$ such that $\langle\kappa, \alpha\rangle+|\alpha|-j>0$, we have

$$b_{j,\alpha}(t, x)=t^{\langle\kappa,\alpha\rangle+|\alpha|-j} c_{j,\alpha}(t, x) \qquad \text{for some } c_{j,\alpha}(t, x) \in \mathcal{B}^\infty(\Omega).$$

Then P satisfies our conditions with $k=0$, $S(t, \xi)=t^{2\kappa_1}\xi_1^2+\cdots+t^{2\kappa_n}\xi_n^2$. If $\kappa_1=\cdots=\kappa_n=0$, then P is nothing but a regularly hyperbolic operator in the sense of Petrowsky.

(5) Let P be the operator

$$P(t, x, \partial_t, \partial_x)=\partial_t^2-\sum_{ij=1}^n t^{\kappa_i+\kappa_j} a_{ij}(t, x)\partial_{x_i}\partial_{x_j}+\text{(lower order terms)}.$$

We assume the following conditions on P:

(i) $\kappa_i=0, \dfrac{1}{2}$ or 1,

(ii) If $\kappa_i+\kappa_j=\dfrac{1}{2}$ or $\dfrac{3}{2}$, then $a_{ij}(t, x) \equiv 0$,

(iii) $\sum_{ij=1}^n a_{ij}(t, x)\xi_i\xi_j \geqq c|\xi|^2$ $(c>0)$.

Then P satisfies our conditions with $m=2$, $k=0$, $S(t, \xi)=t^{2\kappa_1}\xi_1^2+\cdots+t^{2\kappa_n}\xi_n^2$ for any lower order terms. Hence P is a strongly hyperbolic operator. For example, the following operators are of this type:

$$P=\partial_t^2-t\partial_x^2,$$
$$P=\partial_t^2-\partial_{x_1}^2-t\partial_{x_2}^2-t^2\partial_{x_3}^2.$$

(6) Let P be the operator

$$P(t, x, \partial_t, \partial_x)=\partial_t^3-\sum_{ij=1}^n t^{\kappa_i+\kappa_j} a_{ij}(t, x)\partial_{x_i}\partial_{x_j}\partial_t+\text{(lower order terms)}.$$

We assume the following conditions on P:

(i) $\kappa_i = 0$ or $\dfrac{1}{2}$,

(ii) If $\kappa_i + \kappa_j = \dfrac{1}{2}$, then $a_{ij}(t, x) \equiv 0$,

(iii) $\sum_{ij=1}^{n} a_{ij}(t, x)\xi_i \xi_j \geqq c|\xi|^2$ $(c>0)$.

Then P satisfies our conditions with $m=3$, $k=0$, $S(t, \xi) = t^{2\kappa_1}\xi_1^2 + \cdots + t^{2\kappa_n}\xi_n^2$ for any lower order terms. Hence P is a strongly hyperbolic operator. For example, the following operators are of this type:

$$P = \partial_t^3 - t\partial_x^2 \partial_t,$$
$$P = \partial_t^3 - (\partial_{x_1}^2 + t\partial_{x_2}^2)\partial_t.$$

(7) Let P be the operator

$$P(t, x, \partial_t, \partial_x) = \partial_t^2 - \sum_{ij=1}^{n} \partial_{x_i}\partial_{x_j} - t\sum_{i=1}^{n} \partial_{x_i}^2 + (\text{lower order terms}),$$
$$P(t, x, \partial_t, \partial_x) = \partial_t^3 - \left(\sum_{ij=1}^{n} \partial_{x_i}\partial_{x_j} + t\sum_{i=1}^{n} \partial_{x_i}^2\right)\partial_t + (\text{lower order terms}).$$

Then P satisfies our conditions with $m=2$ or 3, $k=0$, $S(t, \xi) = \sum_{ij=1}^{n} \xi_i \xi_j + t\sum_{i=1}^{n} \xi_i^2$ for any lower order terms. Hence P is a strongly hyperbolic operator.

As for necessary conditions for strongly hyperbolicity, see Ivrii-Petkov [4].

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