65. *Amida-Diagrams and Seifert Matrices of Positive Iterated Torus Knots*

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(Communicated by Kunihiko KODAIRA, M. J. A., Oct. 12, 1978)

§1. **Introduction.** In [12], Seifert gave a method to construct an orientable surface which spans a knot by means of a regular projection. As for the torus knot with canonical regular projection, he proved that the surface spanning it has the minimal genus.

Let $\mathcal{T}_q=\{(m_1, \lambda_i)\}_{i=1}^q$ be a sequence of pairs of relatively prime integers. As a generalization of a torus knot, an iterated torus knot $k(\mathcal{T}_q)$ of type $\mathcal{T}_q$ is defined. If all integers are positive and all crossings of $k(\mathcal{T}_q)$ are of the same type, we shall call $k(\mathcal{T}_q)$ to be "*positive*".

In this paper we shall give a representation, *Amida-diagram* of a positive iterated torus knot by means of a certain regular projection. By making use of it, we shall explicitly construct the *Seifert's surface with the minimal genus* and a 1-homology basis of the surface with respect to which *the Seifert matrix is unimodular and lower triangular.*

Our results are related to some properties of irreducible complex analytic curve singularity. (Refer to [3] and [6, Problem 1.5]: for representations of iterated torus knots, [1] and [2]; for singularity, [13], [10] and [8]; and for Seifert matrix, [3] and [7].) Hacon [5] studies the Seifert matrices of iterated torus knots from a different point of view (see also Ohkawa [9]).

The author wishes to thank Prof. M. Kato for some helpful comments and suggestions.

§2. **The regular projections of iterated torus knots.** We consider a knot k in the euclidean 3-space R^3. Let $\pi:\mathrm{R}^3\to\mathrm{R}^2$ be a regular projection of k. Let $\rho:\pi(k)\to\mathrm{R}^3$ be a correspondence such that ρ lifts up the overpaths in small neighborhoods of the double points of $\pi(k)$ and the inclusion map in the outside as in Fig. 1. We call this correspondence $\rho:\pi(k)\to\mathrm{R}^3$ to be the *orthogonal lift* of $\pi(k)$, and denote $\rho\pi(k)$ by K. Obviously k and K are of the same type.

Let $\mathcal{T}_q$ be a sequence of pairs of relatively prime positive integers $\{(m_i, \lambda_i)\}_{i=1}^q$ for $q\geqq 1$ and $\mathcal{T}_j$ be a subsequence $\{(m_i, \lambda_i)\}_{i=1}^j$ for $j=1, \cdots, q$. Then iterated torus knot $k(\mathcal{T}_q)$ of type $\mathcal{T}_q$ is inductively constructed as follows; let $k(\mathcal{T}_0)$ be an unknotted circle oriented counterclockwise in the plane $\mathrm{R}^2 \subset \mathrm{R}^3$. We suppose that $k(\mathcal{T}_{q-1})$ $(q \geqq 1)$ has been constructed. Let T be a tubular neighborhood of $k(\mathcal{T}_{q-1})$ in R^3 with boundary ∂T. Then $k(\mathcal{T}_q)$ is defined as being a knot which lies on ∂T and sweeps around T m_q times in its longitude and λ_q times in its meridian, where orientations of the longitude and the meridian are parallel to $k(\mathcal{T}_{q-1})$ and left-handed screw with respect to $k(\mathcal{T}_{q-1})$ respectively. The orientation of $k(\mathcal{T}_q)$ is induced by that of $k(\mathcal{T}_{q-1})$. Then $k(\mathcal{T}_1)$ is the so-called torus knot of type (m_1, λ_1).

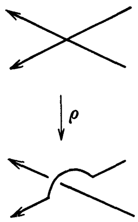

Fig. 1

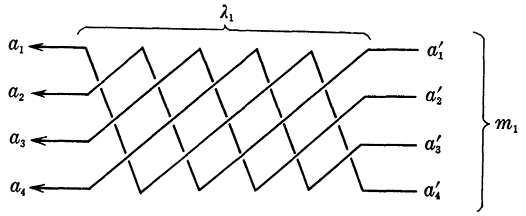

Fig. 2. $K(\mathcal{T}_1)$ of type $\mathcal{T}_1 = \{(4, 5)\}$.

For the torus knot $k(\mathcal{T}_1)$ there is the canonical regular projection $\pi : \mathrm{R}^3 \to \mathrm{R}^2$ such that the double points of $\pi(k(\mathcal{T}_1))$ are arranged $m_1 - 1$ in rows and λ_1 in columns. (See Fig. 2, where $\pi(k(\mathcal{T}_1))$ is obtained by identifying a_i with a'_i for each $i = 1, \cdots, m_1$.)

We consider of the braid representation of the orthogonal lift $K(\mathcal{T}_1)$ of $\pi(k(\mathcal{T}_1))$. We note that all crossings of $K(\mathcal{T}_1)$ are of the same type, that is, at each crossing point the overpath crosses through the underphath from the right to the left with respect to the orientation as in Fig. 2. Let b_i $(i = 1, \cdots, m_1 - 1)$ be a generator of the braid group as in Fig. 3. Let $B(\mathcal{T}_1)$ be the braid representation of $K(\mathcal{T}_1)$. Then we have

$$B(\mathcal{T}_1) = (\alpha^1)^{\lambda_1},$$

where α^1 is a word of the braid group

$$\alpha^1 = b_1 b_2 \cdots b_{m_1 - 1},$$

(see Fig. 4).

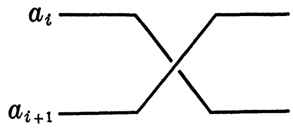

Fig. 3. b_i.

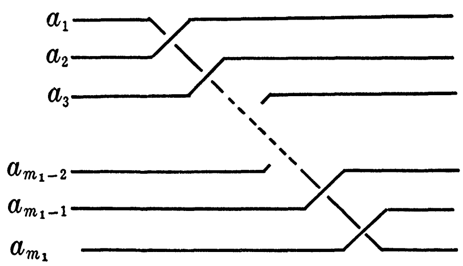

Fig. 4. α^1.

From the construction of the iterated torus knot $k(\mathcal{T}_q)$, we can show in the same way as for $K(\mathcal{T}_1)$ that there is a regular projection $\pi : \mathrm{R}^3 \to \mathrm{R}^2$ of $k(\mathcal{T}_q)$ $(q \geqq 2)$ which is characterized by the braid representation of the orthogonal lift $K(\mathcal{T}_q)$ of $\pi(k(\mathcal{T}_q))$ as follows; let α^j and β_i^j denote the words

$$\alpha^j = b_{(\eta_{j-1}-1)m_j+1} b_{(\eta_{j-1}-1)m_j+2} \cdots b_{\eta_{j-1}m_j-1},$$
$$\beta_i^j = (b_{im_j} \cdots b_{(i+1)m_j-1})(b_{im_j-1} \cdots b_{(i+1)m_j-2}) \cdots (b_{(i-1)m_j+1} \cdots b_{im_j}),$$

for $j=2, \cdots, q$ and $i=1, \cdots, \eta_{j-1}-1$, where

$$\eta_j = m_1 m_2 \cdots m_j.$$

Let n_j and d_j be

$$n_1 = \lambda_1$$
$$n_j = \lambda_j + n_{j-1}m_j - \lambda_{j-1}m_{j-1}m_j \qquad j=1, \cdots, q,$$
$$d_0 = 0,$$
$$d_j = m_j^2 d_{j-1} + (m_j - 1)|n_j| \qquad j=1, \cdots, q.$$

We denote the braid representation of $K(\mathcal{T}_j)$ $(j=1, \cdots, q)$ by $B(\mathcal{T}_j)$. Then we have

Theorem 1. *There is a regular projection $\pi: \mathrm{R}^3 \to \mathrm{R}^2$ of $k(\mathcal{T}_q)$ $(q \geqq 2)$ such that $B(\mathcal{T}_q)$ is given by*

$$B(\mathcal{T}_q) = (B(\mathcal{T}_{q-1}) \underset{i}{\otimes} \beta_i^q)(\alpha^q)^{n_q},$$

where $\otimes_i$ is the operation which substitutes β_i^q into b_i in $B(\mathcal{T}_{q-1})$.

We note that d_j denotes the number of the double points of $\pi(k(\mathcal{T}_j))$ for $j=0, \cdots, q$.

Thus all crossings of $K(\mathcal{T}_q)$ are of the same type when $n_j > 0$ for all $j=1, \cdots, q$. We call $K(\mathcal{T}_q)$ to be *positive* if $n_j > 0$ for all $j=1, \cdots, q$. Since torus knots $K(\mathcal{T}_1)$ are positive, we can simplify the diagram in Fig. 2 as in Fig. 5, where each vertical line connecting two horizontal lines represents the crossing as in Fig. 3. For positive iterated torus knots $K(\mathcal{T}_q)$ we can simplify the diagram of $K(\mathcal{T}_q)$ as in Fig. 6,

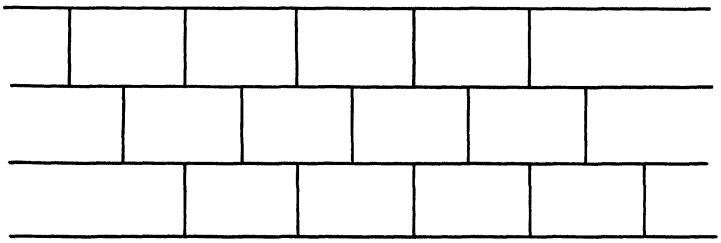

Fig. 5. The Amida-diagram of $K(\mathcal{T}_1)$ of type $\mathcal{T}_1 = \{(4, 5)\}$.

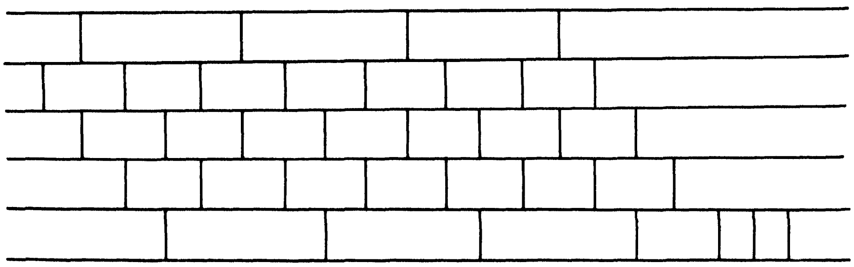

Fig. 6. The Amida-diagram of $K(\mathcal{T}_2)$ of type $\mathcal{T}_2 = \{(3, 4), (2, 19)\}$.

as in case of $K(\mathcal{T}_1)$. We call this simplified diagram of $K(\mathcal{T}_q)$ the "*Amida-diagram*" of $K(\mathcal{T}_q)$.

§3. Seifert's surfaces and Seifert matrices of positive iterated torus knots. Seifert [12] gave a method to construct an orientable surface which spans a knot by means of a regular projection as follows; we start anywhere on the knot and follow it along in the positive direction until we come to a crossing point, then we hop over the other branch and follow it in the positive direction until we come to another crossing point, and so on, until we close up. Then we get a circle, *Seifert circle*. We do this work until there is no part of the knot that we have not passed. Then we get a number of Seifert circles which are disjoint. We span a half-twisted rectangle at each crossing so that the surface consisting of the disks with boundary Seifert circles and the half-twisted rectangles is orientable and spanning the knot.

As for the torus knot $k(\mathcal{T}_1)$, Seifert [12] showed that the Seifert's surface spanning it with respect to the canonical regular projection has the minimal genus.

From now on, we consider positive iterated torus knots $K(\mathcal{T}_q)$. We may think that the Amida-diagram of $K(\mathcal{T}_q)$ represents its Seifert's surface, that is, each horizontal line represents the Seifert circle spanning a disk vertical to the plane and each vertical line represents a half-twisted band. We denote the Seifert's surface of $K(\mathcal{T}_q)$ by $\mathcal{S}_q$.

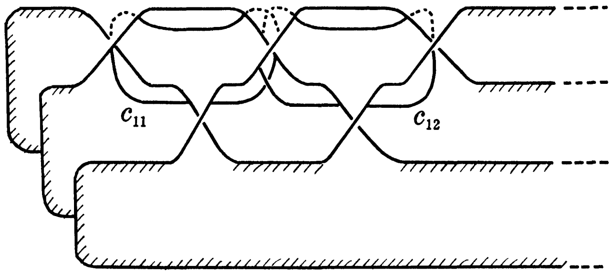

Fig. 7

In the Amida-diagram of $K(\mathcal{T}_q)$, which can be also considered as the Amida-diagram of $\mathcal{S}_q$, we denote the ith $(1 \leqq i \leqq m_1 \cdots m_q)$ horizontal line from the top by F_i and jth vertical line from the left connecting F_i and F_{i+1} by b_i^j. Let c_{ij} be the canonical closed curve on $\mathcal{S}_q$ which passes through b_i^j, F_{i+1}, b_i^{j+1} and F_i as in Fig. 7. By $\langle c_{ij}, c_{kl}{}^{+} \rangle$ we denote the linking number of c_{ij} and c_{kl} which is pushed out of $\mathcal{S}_q$ in a normal direction in R^3. Then by suitable choice of the orientation, we obtain the following.

Lemma. *For the closed curves c_{ij} and c_{kl} on $\mathcal{S}_q$, we have*

$$\langle c_{ij}, c_{kl}^{+}\rangle = \begin{cases} 1 & i=k \text{ and } j=l, \\ -1 & i=k \text{ and } j=l+1, \\ \pm 1 \text{ or } 0 & i=k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma(\mathcal{T}_q)$ be the Seifert matrix of a positive iterated torus knot $K(\mathcal{T}_q)$ with respect to the 1-homology basis $\{c_{ij}\}$ and let

$$g_q = \frac{1}{2}(d_q - m_1 \cdots m_q + 1) \qquad q=1, 2, \cdots.$$

Theorem 2. *$\Gamma(\mathcal{T}_q)$ is the $2g_q \times 2g_q$ lower triangular unimodular matrix.* (Compare with Durfee [3].)

In particular we have

Corollary 1. *$\mathcal{S}_q$ is the surface spanning $K(\mathcal{T}_q)$ with minimal genus g_q.*

If $n_{j-1}m_j < n_j$ for all $j=2, \cdots, q$, we note that $k(\mathcal{T}_q)$ is the knot which is associated with an irreducible polynomial $f(z_1, z_2)$ in two complex variables with $f(0,0)=0$ whose Puiseux characteristic pairs about the origin are $(n_1, m_1), \cdots, (n_q, m_q)$. In this case, from Sakamoto [9] we have

Corollary 2. *A tensor product of matrices $\Gamma(\mathcal{T}_q)$ is realized as being the Seifert matrix of an isolated hyper surface singularity in* $\mathbf{C}^{n+1}$. (Refer to Kato [6, Problem 1.5].)

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