61. *Closedness of q-Ideals in a Compact and Totally Disconnected Semigroup*

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1. A *topological semigroup* is a semigroup with a Hausdoff topology in which multiplication is continuous in both variables. In what follows S will denote a topological semigroup. An ideal P of S is termed *prime* if $AB \subset P$ implies that either $A \subset P$ or $B \subset P$, A and B being ideals of S. The notion of q-ideals has been defined in [6], namely, an ideal of S is called, briefly, a *q-ideal* if it is expressed as an intersection of open prime ideals of S.

Our main objective of this paper is to establish a necessay and sufficient condition for a q-ideal of S to be closed provided that S is compact and totally disconnected. As an application, we shall show that the radical of a compact and totally disconnected topological semigroup with zero is closed.

Throughout the whole paper we shall use the following notation.

A^* denotes the topological closure of a subset A of S.

$X-Y$ means the set of elements of X which are not in Y, where X and Y are any two sets. We write $X-y$ instead of $X-\{y\}$ when $\{y\}$ is a singleton.

E denotes the set of all idempotents in S. E is known to be a closed subset of S, and it is not empty if S is compact.

$J_0(A)$ means the union of all ideals of S which are contained in A, i.e., $J_0(A)$ is the largest ideal contained in A if $J_0(A) \neq \emptyset$, where A is a subset of S.

2. The following lemma is an analogy of the well-known result in the theory of topological groups (e.g. see [2]).

Lemma 2.1. *Let S be locally compact and totally disconnected, and let S have a right* [*or left*] *identity e. Then any neighborhood of e contains a compact subsemigroup neighborhood of e.*

Proof. Let W be any neighborhood of e. Since S is a locally compact and totally disconnected Hausdorff space, there exists a compact and open subset U of S such that $e \in U \subset W$.

Let

$$C=(S-U) \cap U^2,$$

so that C is closed. Since $Ue \cap C$ $(=U \cap C)$ is empty and U is compact, there is an open neighborhood V of e such that $V\subset U$ and $UV\cap C=\emptyset$; since $UV\subset U^2$, this implies that $UV\subset U$. Then, of course, $UV^n\subset U$ for every positive integer n. From $V\subset U$ it follows that $\cup_n V^n\subset U$. Then $T=(\cup_n V^n)^*$ is the desired semigroup neighborhood of e.

For the case where S has a left identity the proof is analogous.

Proposition 2.2. *Let S be as in Lemma 2.1. If W is a neighborhood of e containing no idempotent other than e, then W contains a compact, open group neighborhood of e.*

Proof. By Lemma 2.1, there exists a compact subsemigroup G such that $G\subset W$ and G is a neighborhood of e. G is a compact topological semigroup with the right [or left] identity e, and moreover it has no idempotent distinct from e; therefore from the structure theory of compact semigroups (e.g. see [1, Corollary 2 to Theorem 2]) one can readily see that G is a group. Let $x\in G$. If V is an open neighborhood of e such that $V\subset G$, then xV is an open neighborhood of x with $xV\subset G$. This implies that G is an open subset of S. Thus G is the desired group neighborhood of e.

3. We now proceed to prove the main theorem of this paper.

Theorem 3.1. *Let S be compact and totally disconnected. Then, a q-ideal Q of S is closed in S if and only if $E\cap Q$ is closed in S.*

Proof. The "only if" part is obvious.

To prove the "if" part let us assume that Q is not closed and seek a contradiction. Since Q^* is an ideal of S not contained in Q, by Theorem 2.6 in [6] there is an idempotent e in Q^*-Q. Then there exists a neighborhood U of e such that $U\cap(E\cap Q)=\emptyset$, because $e\notin E\cap Q=(E\cap Q)^*$.

Now consider the semigroup Se. This semigroup is a compact, totally disconnected topological semigroup as a subspace of S. Accordingly we can apply Lemma 2.1 to this semigroup, so that we can find a compact subsemigroup T of Se such that $T\subset U$ and T is a neighborhood of e in the space Se. Hence there exists a neighborhood V of e in the space S such that $V\cap Se\subset T$.

Next we shall show that T does not meet Q. If otherwise, there is an element x in $T\cap Q$. Let f be the idempotent in the closure of the positive powers of x, i.e., $f=f^2\in\{x^n : n=1, 2, \cdots\}^*$. Then f must be contained in T, because T is a compact semigroup. At the same time, it follows from $x\in Q$ that $f\in Q$, since Q is an ideal of S (in this connection, see the proof of [3, Theorem 1]). That is, $f\in E\cap Q$. Therefore we would have $\emptyset\neq T\cap(E\cap Q)\subset U\cap(E\cap Q)$, and this contradiction shows that $T\cap Q=\emptyset$.

It follows therefore that $V\cap Se\cap Q\subset T\cap Q=\emptyset$. From $Se\cap Q=Qe$ we have $V\cap Qe=\emptyset$. On the other have $e\in Q^*$ implies $e\in Q^*e=(Qe)^*$, and this means that $V\cap Qe\neq\emptyset$. Thus we arrived at a contradiction. This completes the proof.

As an immediate consequence of the preceding theorem we obtain the follwing result.

Corollary 3.2. *Let S be compact and totally disconnected, and let Q be a q-ideal of S. If Q contains only a finite number of idempotents, then Q is closed in S.*

Suppose now that S has a zero element, 0. An element b of S is said to be *nilpotent* if $b^n \to 0$. We denote by N_0 the set of all nilpotent elements of S. The largest ideal contained in N_0 is called the *radical* of S and denoted by N, i.e., $N = J_0(N_0)$.

Corollary 3.3. *Let S be compact and totally disconnected, and let S have a zero element. Then the radical of S is closed in S.*

Proof. It has been shown in [4] that N is the intersection of all open prime ideals of S (see [4, Theorem 1]). Therefore N is a q-ideal of S. Furthermore we have $E \cap N = \{0\}$. Applying the preceding corollary we can conclude that N is closed in S.

Combining the above corollary and Corollary 3.5 in [5], we obtain the following result which is a partial answer to the problem described in [5].

Corollary 3.4. *Let S be as in Corollary 3.3. If the radical N is open, then N can be expressed as an intersection of a finite number of open prime ideals.*

4. Let $e, f \in E$. We say that f is *under* e if $f \neq e$ and $ef = f = fe$.

In proving Theorem 4.1 we need the following result which has been proved in [4, Theorem 2]: *Let S be compact and P a proper ideal of S; then P is open and prime if and only if P has the form $P = J_0(S-e)$, $e \in E$.*

Theorem 4.1. *Let S be compact and totally disconnected, and let P be the proper open prime ideal of S with the form $P = J_0(S-e)$, $e \in E$. If we denote by F the set of all idempotents which are under e, then P is closed in S if and only if $e \notin F^*$.*

Proof. The "only if" part: Suppose that P is closed, i.e., $P = P^*$. Let $f \in F$. From $fe = f = ef$, we have $f \in eSe$. Since e is a P-primitive idempotent (see [5, Lemma 2.1]), f must be contained in P, that is, $F \subset P$. (An idempotent g of S is said to be a *P-primitive idempotent* if $g \notin P$ and g is the only idempotent in $gSg - P$.) Hence we obtain $F^* \subset P^* = P \not\ni e$.

The "if" part: Suppose that $e \notin F^*$. We assume the converse, that $P^* \neq P$. Since P^* is an ideal of S properly containing P, P^* must contain e. By the assumption there exists a neighborhood U of e such that $U \cap F = \emptyset$.

Now consider the semigroup eSe. This semigroup is a compact, totally disconnected topological semigroup as a subspace of S. Furthermore eSe has the identity e. Accordingly we can apply Lemma 2.1 to the semigroup eSe, and so there exists a compact subsemigroup T of eSe such that $T\subset U$ and T is a neighborhood of e in the space eSe. Hence there is a neighborhood V of e in the space S such that $V\cap eSe\subset T$.

We shall next show that $T\cap P=\emptyset$. If $T\cap P\neq\emptyset$, then by the same argument as in the proof of Theorem 3.1 we can find an idempotent f in $T\cap P$. From $f\in eSe$ we have $ef=f=fe$, and f is different from e, because $f\in P$. Therefore f is under e, i.e., $f\in F$. Hence we would have $\emptyset\neq T\cap F\subset U\cap F$, and this contradiction shows that $T\cap P=\emptyset$.

It follows therefore that $V\cap eSe\cap P=\emptyset$. From this we see that $V\cap ePe=\emptyset$, since $eSe\cap P=ePe$. On the other hand, it follows from $e\in P^*$ that $e\in eP^*e=(ePe)^*$. And this implies that $V\cap ePe\neq\emptyset$. Thus we arrived at the contradiction, which completes the proof.

Corollary 4.2. *Let S be compact and totally disconnected. If P is an open prime ideal of S which is minimal among the open prime ideals of S, then P is closed in S.*

Proof. We denote by K the kernel (=the unique minimal ideal) of S; K is well-known to be closed. Let us suppose that P has the form $P=J_0(S-e)$, $e\in E$ and denote by F the set of all idempotents which are under e. Then F must be contained in K, because P is minimal among the open prime ideals of S. It follows therefore that $F^*\subset K^*=K\subset P$, and hence $e\notin F^*$. Applying Theorem 4.1, we can conclude that P is closed in S.

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