14. Non-Immersion and Non-Embedding Theorems for Complex Grassmann Manifolds*)

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Introduction. The purpose of the present paper is to prove non-immersion and non-embedding theorems for the complex Grassmann manifolds $G_{k,n-k}=U(n)/U(k)\times U(n-k)$ by making use of an index theorem due to Atiyah-Hirzebruch [1]. We denote by $X\subseteq R^q$ (or $X\subset R^q$) the existence of immersion (or embedding) of a differentiable manifold X into the Euclidean space R^q respectively. Let $\alpha(q)$ denote the number of 1's in the dyadic expansion of an integer q. Then our results are stated as follows:

Main Theorem. *Let $2m=2k(n-k)$ be the dimension of $G_{k,n-k}$ and let $r=\sum_{j=1}^{k}(\alpha(n-j)-\alpha(j-1))$. Then,*

(a) (i) $G_{k,n-k}\not\subset R^{4m-2r}$, (ii) $G_{k,n-k}\not\subseteq R^{4m-2r-1}$.

(b) *If n is odd, then $m=k(n-k)$ is even and*

(i) *if* $r\equiv 3 \pmod 4$ *then* $G_{k,n-k}\not\subset R^{4m-2r+2}$,

(ii) *if* $r\equiv 2$ *or* $3 \pmod 4$ *then* $G_{k,n-k}\not\subseteq R^{4m-2r+1}$,

if $r\equiv 1 \pmod 4$ *then* $G_{k,n-k}\not\subseteq R^{4m-2r}$.

These are generalizations of results for complex projective spaces investigated by Atiyah-Hirzebruch [1], Sanderson-Schwarzenberger [5] and Mayer [4] and of the results for some complex Grassmann manifolds obtained by Sugawara [6].

This paper is arranged as follows. In § 1, the index theorem for immersion and embedding due to Atiyah-Herzebruch [1] and Mayer [4] are recalled. § 2 is devoted to show the computability of some Todd genus for complex homogeneous spaces G/U. We prove Main Theorem in § 3 and exhibit Table I of r for some n and k.

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§ 1. Index theorems. Let X^{2m} be a closed connected oriented differentiable manifold of $2m$ dimension. Let $\{\hat{A}_j(p_1,p_2,\cdots,p_j)\}$ be the multiplicative sequence of polynomials [3, § 1] with $(z/2)/\sinh(z/2)$ as characteristic power series and let $\hat{\mathcal{A}}(X)$ be the cohomology class $\sum_{j=0}^{[m/2]}A_j(p_1(\xi),\cdots,p_j(\xi))$ of the tangent bundle $\xi=\tau(X)$ of X. For any $z\in H^*(X,Q)$ and $d\in H^2(X,Q)$, we define $\hat{A}(X,d,z)=\{ze^d\hat{\mathcal{A}}(X)\}[X]$. Let

*) Dedicated to Professor Atuo Komatu for his 70th birthday.

Table I

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1														
3	1	1													
4	2	2	2												
5	1	2	2	1											
6	2	2	3	2	2										
7	2	3	3	3	3	2									
8	3	4	5	4	5	4	3								
9	1	3	4	4	4	4	3	1							
10	2	2	4	4	5	4	4	2	2						
11	2	3	3	4	5	5	4	3	3	2					
12	3	4	5	4	6	6	6	4	5	4	3				
13	2	4	5	5	5	6	6	5	5	5	4	2			
14	3	4	6	6	7	6	7	6	7	6	6	4	3		
15	3	5	6	7	8	8	7	7	8	8	7	6	5	3	
16	4	6	8	8	10	10	10	8	10	10	10	8	8	6	4
17	1	4	6	7	8	9	9	8	8	9	9	8	7	6	4
18	2	2	5	6	8	8	9	8	9	8	9	8	8	6	5
19	2	3	3	5	7	8	8	8	9	9	8	8	8	7	5
20	3	4	5	4	7	8	9	8	10	10	10	8	9	8	7
21	2	4	5	5	5	7	8	8	9	10	10	9	8	8	7
22	3	4	6	6	7	6	8	8	10	10	11	10	10	8	8
23	3	5	6	7	8	8	7	8	10	11	11	11	11	10	8
24	4	6	8	8	10	10	10	8	11	12	13	12	13	12	11
25	2	5	7	8	9	10	10	9	9	11	12	12	12	12	11

Table of r

$r=\sum_{j=1}^{k}(\alpha(n-j)-\alpha(j-1))$

$ch(X)$ be the subring of $H^*(X, Q)$, the image of Chern character $ch: K(X) \to H^*(X, Q)$. For an element $z=\sum_{i=0}^{m} z_i \in ch(X)$ with $z_i \in H^{2i}(X, Q)$, we write $z^{(t)}=\sum_{i=0}^{m} z_i t^i$ where t is an indeterminate. The Hilbert polynomial $H(t)$ with respect to $z \in ch(X)$ and $d \in H^2(X, Z)$ is defined (Atiyah-Hirzebruch [1, § 2.5]) as follows:

(1.1) $$H(t)=\hat{A}(X, d/2, z^{(t)})=\{z^{(t)} e^{d/2} \hat{\mathscr{A}}(X)\}[X].$$

We denote by $\nu_p(k)$ the (positive or negative) exponent of a prime p in a rational number k, that is, $k=\prod_p p^{\nu_p(k)}$. For the next theorem, see Atiyah-Hirzebruch [1, §2.6], Sanderson-Schwarzenberger [5, Theorem 4] and Mayer [4, § 4.3].

and let $H(t)$ be the Hilbert polynomial with respect to $z \in ch(X)$ and $d \in H^2(X, Z)$. Let $r = 2m + \nu_2(H(1/2))$, then

(i) $X \not\subset R^{4m-2r}$, (ii) $X \not\subseteq R^{4m-2r-1}$.

When the dimension of X is divisible by four, Mayer [4, § 4.3] improved above theorem. Let $chO(X)$ be the subring of $ch(X)$, the image of Chern character composed with the complexification $c; KO(X) \to K(X)$.

Theorem 1.2. *Let $2m$ be the dimension of X with m even and let $H(t) = \hat{A}(X, 0, z^{(t)})$ be the Hilbert polynomial with respect to $z \in chO(X)$ and $d = 0$. Let $r = 2m + \nu_2(H(1/2))$, then*

(i) *if* $r \equiv 3 \pmod 4$ *then* $X \not\subseteq R^{4m-2r+2}$

(ii) *if* $r \equiv 2$ *or* $3 \pmod 4$ *then* $X \not\subseteq R^{4m-2r+1}$

if $r \equiv 1 \pmod 4$ *then* $X \not\subseteq R^{4m-2r}$

If X is moreover endowed with an almost complex structure, then the Todd class $\mathcal{T}(X)$ can be defined as the cohomology class $\sum_{j=0}^{m} T_j(c_1(\xi), \cdots, c_j(\xi))$ of the tangent bundle $\xi = \tau(X)$ where $\{T_j(c_1, \cdots, c_j)\}$ is the Todd multiplicative sequence of polynomial [3, § 1] with $x/(1-e^{-x})$ as its characteristic power series. In this case choosing $d = c_1(X)$ the first Chern class of X, (1.1) is rewritten as

$$H(t) = \{z^{(t)}\mathcal{T}(X)\}[X] \tag{1.2}$$

since $\mathcal{T}(\xi) = \exp(c_1(\xi)/2)\hat{\mathcal{A}}(\xi)$ holds for any complex vector bundle ξ.

§ 2. Complex homogeneous space G/U. Let G be a compact connected Lie group and U its closed subgroup of the centralizer of a torus of G. Then U contains a maximal torus T of G and by H. C. Wang G/U is a homogeneous complex manifold (Borel-Hirzebruch [2, § 13.5]). Let V be the universal covering space of T and $\pi: V \to T$ be the projection. Let $\Gamma = \pi^{-1}(e)$ be the lattice point set where $e \in T$ is the unit element. Then the subgroup of the dual space $V^* = \mathrm{Hom}(V, R)$, consisting of all the functions $\varphi \in V^*$ which takes the value in the integers $Z \subset R$ on Γ, is identified with the cohomology group $H^1(T, Z)$ and the latter is identified with the cohomology group $H^2(G/T, Z)$ by the negative transgression. Hence roots or weights of any representation of G are regarded as elements of $H^2(G/T, Z)$ [2, § 10]. The following two theorems play the essential role in computation of $H(t)$.

Theorem 2.1 (Borel-Hirzebruch [2, § 24.7]). *Let Ψ be a set of roots giving the complex structure of G/U. Let β be a weight orthogonal to the roots of U and $(\beta, \alpha) > 0$ for all $\alpha \in \Psi$, where $(\ ,\ )$ denotes the bilinear form induced from the Killing form. Then the Todd genus $T(G/U, \beta) = \{e^{\beta}\mathcal{T}(G/U)\}[G/U]$ is equal to the dimension of the irreducible representation with the highest weight β.*

Theorem 2.2 (Weyl's dimension formula). *Let V be an irreducible representation space with the highest weight β. Then*

$$\dim V = \prod_{\alpha>0} (\beta+\delta, \alpha) \Big/ \prod_{\alpha>0} (\delta, \alpha) \qquad \delta = \sum_{\alpha>0} \alpha/2.$$

Combining these two theorems, the Hilbert polynomial $H(t)$ with respect to $z=e^{\beta}$ and $d=c_1(X)$ is obtained as

$$H(t) = \prod_{\alpha\in\Psi} (t\beta+\delta, \alpha) \Big/ \prod_{\alpha\in\Psi} (\delta, \alpha) \tag{2.1}$$

where the multiplication runs through over the positive complementary roots $\alpha\in\Psi$, since $(\beta, \alpha)=0$ if α is a root of U by orthogonality.

§ 3. **Complex Grassmann manifolds.** In this section, we give the proof of Main Theorem. Let $U(k)\times U(n-k)\to U(n)\to G_{k,n-k}$ be the natural principal bundle and let $\zeta_1\oplus\zeta_2$ be the associated vector bundle. Denoting $c_i=c_i(\zeta_1)$ (or $c'_i=c_i(\zeta_2)$) the i-th Chern class of ζ_1 (or ζ_2) respectively, we have

$$H^*(G_{k,n-k}, Z) = Z[c_1, c_2, \cdots, c_k, c'_1, \cdots, c'_{n-k}]/J^+$$

where J^+ is an ideal generated by the elements $\{\sum_{i+j=k} c_i c'_j\,;\, k>0\}$.

Let $F(n)$ be the complex flag manifold $U(n)/T^n$ where T^n is a maximal torus of $U(n)$. Let $T^n\to U(n)\to F(n)$ be the natural principal bundle and let $\xi_1\oplus\xi_2\oplus\cdots\oplus\xi_n$ be the associated vector bundle. Denoting $x_i=c_1(\xi_i)$ the first Chern class of ξ_i, we have

$$H^*(F(n), Z) = Z[x_1, x_2, \cdots, x_n]/I^+$$

where I^+ is generated by all symmetric polynomials of positive degree in $x_1, x_2, \cdots, x_n$.

Let $\pi: F(n)\to G_{k,n-k}$ be the natural fibre bundle with the fibre $F(k)\times F(n-k)$. Then $\pi^*: H^*(G_{k,n-k}, Z)\to H^*(F(n), Z)$ is a monomorphism and $\pi^*(c_i)$ (or $\pi^*(c'_i)$) is the i-th elementary symmetric polynomial in $x_1, x_2, \cdots, x_k$ (or in $x_{k+1}, \cdots, x_n$) respectively (Borel-Hirzebruch [2, § 16.2]).

Consider the k-th exterior product $\overset{k}{\wedge}\zeta_1$ of ζ_1. Since ζ_1 is a $U(k)$-bundle, $\overset{k}{\wedge}\zeta_1$ is a line bundle and its first Chern class satisfies

$$\pi^*\Big(c_1\Big(\overset{k}{\wedge}\zeta_1\Big)\Big) = x_1+x_2+\cdots+x_k.$$

The Hilbert polynomial with respect to $z=ch\Big(\overset{k}{\wedge}\zeta_1\Big)=\exp(x_1+\cdots+x_k)$ and $d=c_1(G_{k,n-k})$ is

$$H(t) = \{\exp(t(x_1+\cdots+x_k))\mathcal{T}(G_{k,n-k})\}[G_{k,n-k}]. \tag{3.1}$$

By the elementary Lie algebraic theory, the set of the positive roots of $U(n)$ is $\{e_i-e_j\,;\,1\leqq i<j\leqq n\}$. Note that the elements $e_1, e_2, \cdots, e_n\in V^*$ are identified with $-x_1, -x_2, \cdots, -x_n\in H^2(F(n), Z)$ (see § 2). The bilinear form is given by

$$(e_i, e_j) = \delta_{ij} \quad \text{(Kronecker delta)}. \tag{3.2}$$

Now we are ready to prove Main Theorem. For the part (a), by Theorem 1.1, it is sufficient to show

$$r = 2m+\nu_2\Big(H\Big(\frac{1}{2}\Big)\Big) = \sum_{j=1}^{k} (\alpha(n-j)-\alpha(j-1)). \tag{3.3}$$

By (2.1) and (3.1), $H(1/2)$ is equal to

$$\prod_{k+1\leqq j\leqq n}^{1\leqq i\leqq k}((\beta/2)+\delta,\, e_i-e_j)\Big/\prod_{k+1\leqq j\leqq n}^{1\leqq i\leqq k}(\delta,\, e_i-e_j) \tag{3.4}$$

where $\beta=-(e_1+e_2+\cdots+e_k)$ and $\delta=(\sum_{1\leqq i<j\leqq n}(e_i-e_j)/2$. Since $(\delta, e_i-e_j)=j-i$ and

$$(\beta, e_i-e_j)=\begin{cases}1 & \text{if } 1\leqq i\leqq k \text{ and } k+1\leqq j\leqq n,\\ 0 & \text{otherwise,}\end{cases}$$

hold by (3.2), we get

$$H\left(\frac{1}{2}\right)=\prod_{j=1}^{k}\left(\frac{1}{2}+n-j\right)_{n-k}\Big/\prod_{j=1}^{k}(n-j)_{n-k}$$

where $(s)_q$ denotes the multiplication $s(s-1)\cdots(s-q+1)$ for a real number s and a positive integer q. Note that if s is also an integer with $s\geqq q$, then $(s)_q=s!/(s-q)!$ and making use of well known elementary number theoretical formula $s=\nu_2(s!)+\alpha(s)$, we obtain $\nu_2((s)_q)=q-\alpha(s)+\alpha(s-q)$. Thus we have

$$\nu_2\left(\prod_{j=1}^{k}\left(\frac{1}{2}+n-j\right)_{n-k}\right)=-k(n-k)$$

$$\nu_2\left(\prod_{j=1}^{k}(n-j)_{n-k}\right)=k(n-k)-\sum_{j=1}^{k}(\alpha(n-j)-\alpha(k-j))$$

and hence (3.3) is obtained.

Preceding the proof of the part (b) of Main Theorem, we prepare a lemma. Let $\hat{A}(X)=\hat{A}(X,0,1)=\hat{\mathcal{A}}(X)[X]$ be the $\hat{A}$-genus.

Lemma 3.1. *Let $z=ch\left(\bigwedge^{k}\zeta_1\right)$ and $d=c_1(G_{k,n-k})$. If n is odd, then*

$$\nu_2(\hat{A}(G_{k,n-k}, d/2, z^{(1/2)})=\nu_2(\hat{A}(G_{k,n-k})).$$

Proof. Since $\hat{A}(X)=\hat{\mathcal{A}}(X)[X]=\{\exp(-c_1(X)/2))\mathcal{T}(X)\}[X]$, substituting $X=G_{k,n-k}$, we can use Theorems 2.1 and 2.2 again. Note that the first Chern class $c_1(G_{k,n-k})$ is obtained as $-nc_1$ by Borel-Hirzebruch [2, §16.2] where $\pi^*(c_1)=-(x_1+\cdots+x_k)\in H^2(F(n),Z)$. Hence $\hat{A}(G_{k,n-k})$ is equal to (3.4) with $\beta=n(e_1+e_2+\cdots+e_k)$ and the lemma follows.

Now (b) follows from Lemma 3.1 and Theorem 1.2 with

$$z=1\in chO(G_{k,n-k}).$$

References

[1] M. F. Atiyah and F. Hirzebruch: Quelque théorèmes de non-plongement pour les variétés différentiables. Bull. Soc. Math. France, **87**, 383–396 (1959).

[2] A. Borel and F. Hirzebruch: Characteristic classes and homogeneous spaces I, II, III. Amer. J. Math., **80**, 458–538 (1958); **81**, 315–382 (1959); **82**, 491–504 (1960).

[3] F. Hirzebruch: Topological Methods in Algebraic Geometry. 3rd ed., Springer (1966).

[4] K. H. Mayer: Elliptische Differential operatoren und Ganzzahligkeitssätze für Charakteristische Zahlen. Topology, **4**, 295–313 (1965).

[5] B. J. Sanderson and R. L. E. Schwarzenberger: Non-immersion theorems

for differentiable manifolds. Proc. Camb. Phil. Soc., **59**, 319–322 (1963).
[6] T. Sugawara: Immersion and embedding problem for complex flag manifolds. Osaka J. Math., **15**, 117–135 (1978).