67. A Generalization of a Theorem of Marotto*)

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1979)

In 1975, Li and Yorke found the following theorem [3]. Let $f: I \to I$ be a continuous map of the compact interval I into itself. If f has a periodic point of minimal period three, then f exhibits chaotic behavior. This result is generalized by F. R. Marotto [4] in 1978 for the multi-dimensional case as follows. Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a differentiable map of the n-dimensional Euclidean space $\boldsymbol{R}^n$ $(n \geqq 1)$ into itself. If f has a snap-back repeller, then f exhibits chaotic behavior.

In this paper, we shall announce a generalization of the above theorem of Marotto. Our theorem can also be regarded as a generalization of the Smale's result [6] on the transversal homoclinic point of a diffeomorphism. A detailed proof will appear later.

§1. The main theorem. Let M be a smooth manifold of dimension n. Let $f: M \to M$ be a C^1-map, and let $z_0 \in M$ be a hyperbolic fixed point of f. We denote by $W^u_{\mathrm{loc}}(z_0)$ (resp. $W^s_{\mathrm{loc}}(z_0)$) a local unstable (resp. stable) manifold of f at z_0.

Main Theorem. *Let $f: M \to M$ be a C^1-map. Let $z_0 \in M$ be a hyperbolic fixed point of f. Assume the following three conditions.*

(1) $u = \dim W^u_{\mathrm{loc}}(z_0) > 0$.

(2) *There exist a point $z_1 \in W^u_{\mathrm{loc}}(z_0)$ $(z_1 \neq z_0)$ and a positive integer m such that $f^m(z_1) \in W^s_{\mathrm{loc}}(z_0)$.*

(3) *There exists a u-dimensional disk B^u embedded in $W^u_{\mathrm{loc}}(z_0)$ such that B^u is a neighborhood of z_1 in $W^u_{\mathrm{loc}}(z_0)$, $f^m|B^u: B^u \to M$ is an embedding, and $f^m(B^u)$ intersects $W^s_{\mathrm{loc}}(z_0)$ transversally at $f^m(z_1)$.*

Then the following conclusions hold.

(a) *There is a positive integer N such that there is a periodic point of f of minimal period p for any integer $p \geqq N$.*

(b) *There is an uncountable set S (called a scrambled set) in M satisfying the following conditions.*

(i) *S does not contain any periodic points.*

(ii) $f(S) \subset S$

(iii) $\limsup_{k \to \infty} d(f^k(x), f^k(y)) > 0$ *for any $x, y \in S$ $(x \neq y)$, where d is a compatible metric on M.*

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*) Dedicated to Professor A. Komatu on his 70th birthday.

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(iv) $\limsup_{k\to\infty} d(f^k(x), f^k(y))>0$ *for any* $x \in S$ *and a periodic point* y.

(v) *There exists an uncountable subset* S_0 *of* S *such that* $\liminf_{k\to\infty} d(f^k(x), f^k(y))=0$ *for* $x, y \in S_0$.

Remark 1. The above theorem holds if $f: M \to M$ is of class C^1 on a neighborhood of z_0 and on a neighborhood of the orbit of z_1.

Remark 2. In the above theorem the tangent map $T_{z_1}f^m$ of f^m at $z_1 \in M$ may be degenerate.

Remark 3. In case $u=\dim M$, $f^m(z_1)=z_0$ and the above theorem reduces to the theorem of Marotto.

Remark 4. If f is a diffeomorphism with $f^m(z_1)\neq z_0$, then the above assumption implies that $f^m(z_1)$ is a transversal homoclinic point. Thus, our theorem generalizes Smale's result [6] in some sense.

Remark 5. Transversality condition (3) of the main theorem is necessary for the existence of the scrambled set.

§ 2. **Sketch of a proof.** We denote by $T(N)$ the tangent space of a manifold N. The tangent map of a C^1-map $f: N_1 \to N_2$ is denoted by $Tf: T(N_1) \to T(N_2)$.

Let $s=\dim W^s_{\mathrm{loc}}(z_0)=n-u$. Denote by $E^s(r)$ (resp. $E^u(r)$) the s-dimensional (resp. u-dimensional) disk with center at the origin and radius $r>0$. By the stable manifold theorem, we can identify a neighborhood of z_0 with $E^s(r_1)\times E^u(r_1)$ $(r_1>0)$ such that $E^s(r_1)$ (resp. $E^u(r_1)$) is identified with $W^s_{\mathrm{loc}}(z_0)$ (resp. $W^u_{\mathrm{loc}}(z_0)$). Let $\pi^\sigma: E^s(r_1)\times E^u(r_1) \to E^\sigma(r_1)$ $(\sigma=s$ or $u)$ be a canonical projection. Since $T(E^s(r_1)\times E^u(r_1)) = T(E^s(r_1))\times T(E^u(r_1))$, $v \in T(E^s(r_1)\times E^u(r_1))$ can be expressed uniquely as $v=(v^s, v^u)$, $v^\sigma \in T(E^\sigma(r_1))$.

Let r be any positive number smaller than r_1.

Main Lemma. *Assume the same conditions of the main theorem. Let* B^u *be a* u*-dimensional disk in* $E^u(r_1)$, *and let* B^s *be an arbitrary* s*-dimensional disk with the origin* 0. *If* $\psi: B^s\times B^u \to E^s(r_1)\times E^u(r_1)$ *is an embedding such that* $\psi|0\times B^u$ *is the inclusion map* $B^u \subset E^u(r_1)$, *then for any* $\varepsilon>0$ *and* $L>0$ *there is a positive integer* $N(\psi, \varepsilon, L)$ *satisfying the following condition.*

For any integer $n \geqq N(\psi,\varepsilon,L)$, *there is an embedding* $\phi=\phi(\psi,\varepsilon,L,n)$: $E^s(r)\times B^u \to E^s(r_1)\times E^u(r_1)$ *satisfying the following eight conditions.*

(1) $\phi(E^s(r)\times y) \subset \psi(B^s\times y)$ *for* $y \in B^u$.

(2) $f^{-n}(\phi(E^s(r)\times B^u)) \subset E^s(r)\times E^u(r)$.

(3) $f^{-n}(\phi(\partial E^s(r)\times B^u)) \subset \partial E^s(r)\times E^u(r)$, *where* $\partial E^s(r)$ *is the boundary of* $E^s(r)$.

(4) $\pi^s f^{-n}\phi(x\times B^u)=x$ *for* $x \in E^s(r)$.

(5) $\|v^u\|<\varepsilon\|v^s\|$ *for any non-zero* v *in* $T(f^{-n}\phi(E^s(r)\times y))$, $y \in B^u$.

(6) $\|(Tf^{-n}(v))^s\|>L\|v^s\|$ *for any non-zero* v *in* $T(\phi(E^s(r)\times y))$, $y \in B^u$.

(7) $\|(Tf^n(v))^s\| < \varepsilon \|(Tf^n(v))^u\|$ *and*

(8) $\|(Tf^n(v))^u\| > L\|v^u\|$ *for any non-zero* $v \in T(f^{-n}\phi(x \times B^u))$, $x \in E^s(r)$.

This lemma is proved by a similar argument as in the proof of λ-lemma of Palis [5].

Using the above lemma, we can prove the following

Lemma. *Under the assumption of the main theorem, there is a positive integer* N_1 *satisfying the following conditions.*

For any integer $N_0 \geqq N_1$, *there are two embeddings*

$$\phi_i : (E^s(r) \times E^u(r_1), E^s(r) \times B_i^u) \to E^s(r_1) \times E^u(r_1), \qquad (i=0,1)$$

of a pair of rectangles, where B_i^u *is a u-dimensional disk contained in the interior of* $E^u(r_1)$, *satisfying the following ten conditions.*

(1) $f^{N_i}(\phi_i(E^s(r) \times B_i^u)) \subset \phi_j(E^s(r) \times E^u(r_1))$ $(i, j=0,1)$.

(2) $f^{N_i}(\phi_i(E^s(r) \times \partial B_i^u)) \subset \phi_j(E^s(r) \times (E^u(r_1) - B_j^u))$ $(i, j=0,1)$, *where* ∂B_i^u *is the boundary of* B_i^u *in* $E^u(r_1)$.

(3) $f_*^{N_i} : H_{u-1}(\phi_i(E^s(r) \times \partial B_i^u)) \to H_{u-1}(\phi_j(E^s(r) \times (E^u(r_1) - B_j^u)))$ *is an isomorphism, where* $H_{u-1}(\ )$ *is the* $(u-1)$*-th homology group and* $f_*^{N_i}$ *is the induced homomorphism of* f^{N_i} $(i, j=0,1)$.

(4) $\pi^s\phi_i(x \times B_i^u)$ *consists of a single point for* $x \in E^s(r)$, *and* $\pi^s\phi_i(E^s(r) \times B_i^u) = E^s(r)$, $(i=0,1)$.

(5) $\pi^u f^{N_i}\phi_i(E^s(r) \times y)$ *consists of a single point for* $y \in B_i^u$ $(i=0,1)$.

(6) $2\|v^s\| < \|v^u\|$ *for any non-zero* v *in* $T(f^{N_i}\phi_i(x \times B_i^u))$, $x \in E^s(r)$ $(i=0,1)$.

(7) $2\|v^u\| < \|v^s\|$ *for any non-zero* v *in* $T|\phi_i(E^s(r) \times y)$, $y \in B_i^u$ $(i=0,1)$.

(8) $\|(Tf^{N_i}(v))^u\| > 8\|v^u\|$ *for any non-zero* v *in* $T(\phi_i(x \times B_i^u))$, $x \in E^s(r)$ $(i=0,1)$.

(9) $8\|(Tf^{N_i}(v))^s\| < \|v^s\|$ *for any non-zero* v *in* $T(\phi_i(E^s(r) \times y))$, $y \in B_i^u$ $(i=0,1)$.

(10) $\phi_0(E^s(r) \times B_0^u) \cap \phi_1(E^s(r) \times B_1^u) = \phi$.

Now, define $A_i = \phi_i(E^s(r) \times B_i^u)$, $i=0,1$. Let $\Sigma = \{A_0, A_1\}^Z$ be a two-sided shift on two symbols A_0 and A_1, and let $\sigma : \Sigma \to \Sigma$ be the shift map.

An element of Σ is a bisequence $\underline{a} = (a_i)_{i \in Z}$ such that $a_i = A_0$ or A_1. For $\underline{a} = (a_i)_{i \in Z} \in \Sigma$, define $k(\underline{a}, i) = N_0$ if $a_i = A_0$ and $k(\underline{a}, i) = N_1$ if $a_i = A_1$. Define a subset $F^{-i}(\underline{a})$ of M as follows.

$$F^{-i}(\underline{a}) = \begin{cases} (f|a_0)^{-k(\underline{a},0)} \circ \cdots \circ (f|a_{i-1})^{-k(\underline{a},i-1)}(a_i) & \text{if } i>0 \\ a_0 & \text{if } i=0 \\ f^{k(\underline{a},-1)}(\cdots f^{k(\underline{a},i+1)}(f^{k(\underline{a},i)}(a_i) \cap a_{i+1}) \cap \cdots) \cap a_{-1}) \cap a_0 & \text{if } i<0, \end{cases}$$

Proposition. (a) $\bigcap_{i \in Z} F^{-i}(\underline{a})$ *consists of a single point of* M *for each* $\underline{a} \in \Sigma$.

(b) *A map* $p : \Sigma \to M$ *defined by* $p(\underline{a}) = \bigcap_{i \in Z} F^{-i}(\underline{a})$ *is continuous.*

(c) *If there exists an integer* $i \geqq 0$ *such that* $a_i \neq b_i$ *for* $\underline{a} = (a_i)_{i \in Z}$,

$\underline{b}=(b_i)_{i\in Z}$, *then* $p(\underline{a})\neq p(\underline{b})$.

(d) *If* $N_0=N_1$, *then* $p\circ\sigma=f^{N_0}\circ p$.

Now consider the case $N_0=N_1$. Then $p:\Sigma\to M$ satisfies the following conditions.

(2.1) $p:\Sigma\to M$ is continuous, and $p(\underline{a})\neq p(\underline{b})$ if $a_i\neq b_i$ for some $i\geqq 0$.

(2.2) $f^{N_0}\circ p=p\circ\sigma$.

(2.3) $A_0\cap A_1=\phi$.

Now, we can prove the conclusion (b) of Main Theorem using (2.1), (2.2), and (2.3). Our proof is similar to the one in Li and Yorke [3] and Marotto [4].

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