81. *An Extension of the Aumann-Perles' Variational Problem*[*)]

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1. Introduction. Let $u:[0,1]\times \boldsymbol{R}^l_+ \to \boldsymbol{R}$, $x:[0,1]\to \boldsymbol{R}^l_+$ and consider the following problem:

$$\underset{x}{Maximize} \int_0^1 u(t, x(t))dt$$
$$subject\ to$$
$$\int_0^1 x(t)dt = (1, 1, \cdots, 1).$$

($\boldsymbol{R}^l_+$ designates the non-negative orthant of $\boldsymbol{R}^l$.) The variational problem of this type has a lot of interesting applications to economic analysis (cf. Aumann-Shapley [3], Kawamata [7], and Yaari [9]). Aumann-Perles [2] first examined this problem and established a set of sufficient conditions which assures the existence of an optimal solution. Berliocchi-Lasry [4] and Artstein [1] generalized the problem and proved the existence of solutions respectively in quite different ways.

In this paper, I am going to get a further extension of the problem, the application of which can be seen in recent formulations of welfare economics (cf. Kawamata [7]).

2. An extension of the problem. Let T be a compact metric space, and $\bar{\mu}$ be a non-atomic, positive Radon measure on T with $\bar{\mu}(T) = C < +\infty$. We designate by $\mathfrak{M}_{\bar{\mu}}$ the set of all positive Radon measures μ on T such that

$$\text{(i)}\quad \mu \ll \bar{\mu} \qquad \text{(ii)}\quad \mu(T) \leqq C.$$

Let X be a locally compact Polish space, and let

$$u: T\times X \to \boldsymbol{R}$$
$$g_i: T\times X \to \bar{\boldsymbol{R}}_+ \quad ; \quad i=1, 2, \cdots, l.$$

Then our problem is:

$$\underset{\mu, x}{Maximize} \int_T u(t, x(t))d\mu$$
$$subject\ to$$

(I)
$$\text{a)}\quad \int_T g_i(t, x(t))d\mu \leqq \omega_i \quad ; \quad i=1, 2, \cdots, l$$
$$\text{b)}\quad \mu \in \mathfrak{M}_{\bar{\mu}}$$

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c) $x: T \to X$ is measurable

where $(\omega_1, \omega_2, \cdots, \omega_l)$ is a fixed vector.

$\mu \in \mathfrak{M}_{\bar{\mu}}$ and $x: T \to X$ determine the disintegration of the form:

$$(*) \qquad \gamma = \int_T \delta_t \otimes \delta_{x(t)} d\mu.$$

Hence our problem is equivalent to the problem:

$$\text{(II)} \quad \begin{aligned} &\underset{\gamma}{Maximize} \int_{T \times X} u(t, x) d\gamma \\ &\quad subject\ to \\ &\qquad \text{a)} \int_{T \times X} g_i(t, x) d\gamma \leqq \omega_i \quad ; \quad i = 1, 2, \cdots, l \\ &\qquad \text{b)} \ \gamma \text{ is of the form } (*). \end{aligned}$$

I am indebted to Berliocchi-Lasry [4] for such a transformation of the original problem (I) into the form (II) and a full use of disintegration theory in this problem. In comparison to Berliocchi-Lasry [4], where μ is always fixed, we regard μ as one of the control variables as well as x.

3. Disintegration of measures. Let γ be a Radon measure on $T \times X$ which can be expressed as

$$\gamma = \int_T \delta_t \otimes \nu[t] d\mu(t),$$

where δ_t is the Dirac measure at t, μ is a Radon measure on T, and $\nu: t \mapsto \nu[t]$ is a weak*-measurable mapping on T into the set of all Radon probability measures on X. If such a expression is possible, γ is said to have a *μ-disintegration.* We designate by $\Delta(\mu)$ the set of all Radon measures on $T \times X$ that have μ-disintegrations, and put

$$\Delta(\mathfrak{M}_{\bar{\mu}}) = \bigcup_{\mu \in \mathfrak{M}_{\bar{\mu}}} \Delta(\mu).$$

It may be convenient to collect here a few results on disintegration of measures which are useful in later discussions.

T and X are assumed to be compact throughout this section.

Proposition 1 (Castaing [5]). *Let $\Gamma: T \twoheadrightarrow X$ be a measurable multi-valued mapping such that $\Gamma(t) \subset X$ is compact for all $t \in T$. Then a Radon measure γ on $T \times X$ has a disintegration of the form:*

$$\begin{cases} \gamma = \int_T \delta_t \otimes \nu[t] d\mu \\ \operatorname{supp} \nu[t] \subset \Gamma(t) \qquad a.e.\ (t) \end{cases}$$

if and only if

$$\int_{T \times X} f(t, x) d\gamma \leqq \int_T \sup_{x \in \Gamma(t)} f(t, x) d\mu$$

for all $f \in C(T \times X)$, the set of all continuous real-valued functions on $T \times X$.

Proposition 2 (Maruyama [8]). *Consider*

$$\gamma_n=\int_T \delta_t\otimes\nu_n[t]d\mu_n \quad ; \quad n=1,2,\cdots$$
$$\gamma=\int_T \delta_t\otimes\nu[t]d\mu.$$

(i) *If*

a) $w^*\text{-}\lim \mu_n=\mu$

b) $t_p\to t$ *implies* $w^*\text{-}\lim \nu_n[t_p]=\nu_n[t]$ *for all* n

⟨*continuity*⟩

c) $w^*\text{-}\lim \nu_n[t]=\nu[t]$ *for all* $t\in T$,

⟨*pointwise convergence*⟩

then $w^*\text{-}\lim \gamma_n=\gamma$.

(ii) $w^*\text{-}\lim \gamma_n=\gamma$ *implies* a). *But* b) *and* c) *are not necessarily true.*

Proposition 3. *$\Delta(\mathfrak{M}_{\bar{\mu}})$ is weak*-compact and convex.*

4. Positive normal integrands. A function $g:T\times X\to\bar{\boldsymbol{R}}_+$ is called a *positive normal integrand* (PNI) if there exists a function $h:T\times X\to\bar{\boldsymbol{R}}_+$ such that

(i) h is (Borel) measurable,

(ii) $h(t,x)$ is lower semi-continuous in x for $\bar{\mu}$-almost every t,

(iii) $h(t,\cdot)=g(t,\cdot)$ for $\bar{\mu}$-almost every t.

The following lemma can easily be proved.

Lemma 1. *If T and X are compact and g is a* PNI, *then the mapping*

$$\gamma\mapsto\int_{T\times X} g(t,x)d\gamma$$

is lower semi-continuous on $\Delta(\mathfrak{M}_{\bar{\mu}})$.

Let $g_1, g_2,\cdots,g_l$ be PNI's and let $\Delta(\mathfrak{M}_{\bar{\mu}}; g_1,g_2,\cdots,g_l)$ be the set of all $\gamma\in\Delta(\mathfrak{M}_{\bar{\mu}})$ such that

$$\int_{T\times X} g_i(t,x)d\gamma\leqq\omega_i \qquad \text{for all } i=1,2,\cdots,l.$$

If T and X are compact, then we can conclude, from Lemma 1, that $\Delta(\mathfrak{M}_{\bar{\mu}}; g_1,g_2,\cdots,g_l)$ is weak*-compact.

We can extend this result to the case where X is locally compact.

Proposition 4. *Let T be compact, X be locally compact, and $\tilde{X}=X\cup\{\infty\}$ be the one-point compactification of X. If*

$$g(t,x)=\sum_{i=1}^{l} g_i(t,x)\to+\infty \ (a.e.\ \bar{\mu}) \qquad \text{as } x\to\infty,$$

then $\Delta(\mathfrak{M}_{\bar{\mu}}; g_1,g_2,\cdots,g_l)$ is weak-compact and convex.*

5. Existence of optimal solutions. Proposition 5. *Assume the following three conditions for $u:T\times X\to\boldsymbol{R}$.*

(i) *u is Borel measurable,*

(ii) *$u(t,x)$ is upper semi-continuous in x for $\bar{\mu}$-almost every t,*

(iii) *for any $\varepsilon>0$, there exists a $b_\varepsilon\in L^\infty(\bar{\mu})$ such that*

$$u^+(t,x)\geqq b_\varepsilon(t)\Rightarrow u^+(t,x)\leqq\varepsilon g(t,x)$$

where $u^+(t, x) = \mathrm{Max}\,\{u(t, x), 0\}$.

Then the mapping

$$\gamma \mapsto \int_{T\times X} u(t, x)d\gamma$$

is upper semi-continuous on $\Delta\,(\mathfrak{M}_{\bar{\mu}}\,;\, g_1, g_2, \cdots, g_l)$.

By Propositions 4 and 5, the following problem (A) has a solution.

$$\text{(A)} \qquad \underset{\gamma}{\mathit{Maximize}} \int_{T\times X} u(t, x)d\gamma \qquad \text{on } \Delta\,(\mathfrak{M}_{\bar{\mu}}\,;\, g_1, g_2, \cdots, g_l).$$

Let

$$\gamma^* = \int_T \delta_t \otimes \nu^*[t] d\mu^*$$

be a solution of (A). Then γ^* is obviously a solution of the problem:

$$\text{(B)} \qquad \underset{\gamma}{\mathit{Maximize}} \int_{T\times X} u(t, x)d\gamma \qquad \text{on } \Delta\,(\mu^*\,;\, g_1, g_2, \cdots, g_l).$$

Remark. $\Delta\,(\mu^*\,;\, g_1, g_2, \cdots, g_l)$ is also weak*-compact and convex. See Berliocchi-Lasry [4].

In order to approach our final goal, we have to prepare a couple of results from convex analysis. Proposition 6 comes from Carathéodory's theorem, and Proposition 7 is an easy corollary of Ljapunov's convexity theorem. For the detailed proofs, see Berliocchi-Lasry [4].

Proposition 6. *Let* $\mathfrak{X}$ *be a locally convex topological linear space and* K *be a compact, convex subset of* $\mathfrak{X}$. *Let* $\varphi_i : \mathfrak{X} \to \boldsymbol{R}$ $(i=1, 2, \cdots, l)$ *be affine functions and define*

$$H = \{x \in K \mid \varphi_i(x) \leqq 0\,;\, i=1, 2, \cdots, l\}.$$

Then any extreme point of H *can be expressed as a convex combination of at most* $(l+1)$ *extreme points of* K.

Proposition 7. *Let* μ *be a finite non-atomic measure on* T *and consider the formulas:*

$$\sum_{j=1}^{p} \lambda_j \int_T f_{ij}(t) d\mu \quad ; \quad i=1, 2, \cdots, n$$

$$\lambda_j \geqq 0, \qquad \sum_{j=1}^{p} \lambda_j = 1.$$

Then there exists a decomposition $T_1, T_2, \cdots, T_p$ *of* T *such that*

$$\sum_{j=1}^{p} \lambda_j \int_T f_{ij}(t) d\mu = \sum_{j=1}^{p} \int_{T_j} f_{ij}(t) d\mu \quad ; \quad i=1, 2, \cdots, n.$$

Since the mapping $\gamma \mapsto \int_{T\times X} u(t, x)d\gamma$ is linear and $\Delta\,(\mu^*\,;\, g_1, g_2, \cdots, g_l)$ is convex, γ^* can be assumed to be an extreme point of $\Delta\,(\mu^*\,;\, g_1, g_2, \cdots, g_l)$ without loss of generality. Hence by Proposition 6, there exist measurable mappings $x_j : T \to X$ $(j=1, 2, \cdots, l+1)$

$$\gamma^* = \sum_{j=1}^{l+1} \lambda_j \int_T \delta_t \otimes \delta_{x_j(t)} d\mu^*$$

$$\lambda_j \geqq 0, \qquad \sum_{j=1}^{l+1} \lambda_j = 1.$$

By Proposition 7, there exists a decomposition $T_1, T_2, \cdots, T_{l+1}$ of T such that

$$\int_{T\times X} u(t,x)d\gamma^* = \sum_{j=1}^{l+1}\int_{T_j} u(t,x_j(t))d\mu^*$$

$$\int_{T\times X} g_i(t,x)d\gamma^* = \sum_{j=1}^{l+1}\int_{T_j} g_i(t,x_j(t))d\mu^* \quad ; \quad i=1,2,\cdots,l.$$

If we define

$$x^*(t) = \sum_{j=1}^{l+1} \chi_{T_j}(t)x_j(t),$$

then (μ^*, x^*) is a solution of our problem (I), where $\chi_{T_j}(t)$ is the characteristic function of T_j. The idea of constructing $x^*(t)$ by using Propositions 6 and 7 is completely due to Berliocchi-Lasry [4].

Summing up, we have

Theorem. *Assume the followings:*

a) $u: T\times X \to \boldsymbol{R}$ *satisfies the conditions* (i), (ii) *and* (iii) *in Proposition* 5;

b) $g_i: T\times X \to \bar{\boldsymbol{R}}_+$ $(i=1,2,\cdots,l)$ *is a* PNI *such that* $g(t,x) = \sum_{i=1}^{l} g_i(t,x) \to +\infty$ *(a.e.* μ*) as* $x\to\infty$.

Then our problem (I) *has a solution.*

References

[1] Z. Artstein: On a variational problem. J. Math. Anal. Appl., **45**, 404–415 (1974).

[2] R. J. Aumann and M. Perles: A variational problem arising in economics. Ibid., **11**, 488–503 (1965).

[3] R. J. Aumann and L. S. Shapley: Values of Non-Atomic Games. Princeton University Press, Princeton (1974).

[4] H. Berliocchi and J. M. Lasry: Integrandes et mesures paramétrées en calcul des variations. Bull. Soc. Math. France, **101**, 129–184 (1973).

[5] C. Castaing: Application d'un théorème de compacité à un résultat de désintégration des mesures. C. R. Acad. Sc. Paris, **273**, 1056–1059 (1971).

[6] ——: Sur une nouvelle extension du théorème de Ljapunov. Ibid., **264**, 333–336 (1967).

[7] K. Kawamata: Marginal contribution of a firm and optimal entry (1978) (mimeographed).

[8] T. Maruyama: A note on the disintegration of measures: A convergence theorem (to appear).

[9] M. E. Yaari: On the existence of an optimal plan in continuous-time allocation process. Econometrica, **32**, 576–590 (1964).