# 36. *Multi-Dimensional Generalizations of the Chebyshev Polynomials. I**)

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1. **Introduction.** This paper continues the study of the classes of polynomials in 2 variables given in Dunn and Lidl [3] and generalizes these polynomials in two ways: They are generalized to polynomials in $k$ variables over an arbitrary field $K$; secondly a parameter $b \in K$ is introduced for these polynomials, similar to the generalization of the classical Chebyshev polynomials in one variable as in Dickson [1] and Schur [14]. In analysis, the most important case, of course, is $K=C$ and $b=1$, which gives a natural generalization of the Chebyshev polynomials, see Koornwinder [8]. However, there are also some interesting algebraic and number theoretic properties in the more general case of a field $K$ and $b \in K$, particularly for $K=GF(q)$ the one-dimensional polynomials have been studied extensively; see Lansch and Nöbauer [9], Fried [6] and Schur [14]. We use the same notation as in [3] and obtain generating functions and recurrence relations for generalized Chebyshev polynomials of the first and second kind in $k$ variables. In the present paper we are not considering any of the analytic properties of the polynomials (for $k=1$ see Rivlin [13] or Szegö [15]), such as partial differential operators or orthogonality. A different approach to give multi-dimensional extensions of Chebyshev polynomials is introduced by Hays [7]. For some properties of special functions in $k$ variables and a bibliography including the earlier papers on the subject we refer to [5]. We have organized the presentation of the material into I and II, each consisting of two sections: § 2 Definitions, § 3 Results in I and § 4 Proofs, § 5 Outlook in II.

2. **Definitions.** Dickson [1] generalized the classical Chebyshev polynomials in the following way. Let $K$ be a field, $r(z)=z^2-xz+b$ a polynomial over $K$ with roots $u$ and $v$ in a suitable extension field $L$ of $K$ (e.g. if $K=C$ then $L=C$, if $K=GF(q)$ then $L=GF(q^2)$). Then generalizations of the Chebyshev polynomials in one variable of the first and second kind are given by (2.1) and (2.2), respectively.

(2.1) $$P_n^{-1/2}(x\,;b)=u^n+v^n, \qquad \text{for } n \in Z$$

(2.2) $$P_n^{1/2}(x\,;b)=(u-v)^{-1}(u^{n+1}-v^{n+1}), \qquad \text{for } n \geq 0,$$

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$$P_n^{1/2}(x\,;b)=(u^{-1}-v^{-1})^{-1}(u^{-(n+1)}-v^{-(n+1)}),\qquad \text{for } n<0,$$

where $x=u+v$, $uv=b\in K$. In the special case $K=L=C$, $b=1$, and $n\geq 0$, let $u=e^{i\theta}$, $v=e^{-i\theta}$. Then we have

$$P_n^{-1/2}(2\cos\Theta\,;1)=2T_n(\cos\Theta)=2\cos n\Theta$$

and

$$P_n^{1/2}(2\cos\Theta\,;1)=U_n(\cos\Theta)=(\sin\Theta)^{-1}\sin(n+1)\Theta,$$

where $T_n$ and $U_n$ denote the classical Chebyshev polynomials in one variable of the first and second kind, respectively. With $x=\cos\Theta$ this gives the simple relationship

$$P_n^{-1/2}(2x\,;1)=2T_n(x)\quad\text{and}\quad P_n^{1/2}(2x\,;1)=U_n(x). \tag{2.3}$$

More generally (see Lausch and Nöbauer [9], p. 209, Schur [14]):

$$P_n^{-1/2}(x\,;b)=2(\sqrt{b})^nT_n\left(\frac{x}{2\sqrt{b}}\right) \tag{2.4}$$

$$P_n^{1/2}(x\,;b)=(\sqrt{b})^nU_n\left(\frac{x}{2\sqrt{b}}\right). \tag{2.5}$$

Therefore the polynomials defined in (2.1) and (2.2) can be regarded as generalized Chebyshev polynomials. Now we consider the $k$-dimensional case. Let $u_i$, $1\leq i\leq k+1$, be elements in a suitable extension field $L$ of the field $K$, for example, in case $K=C$ we take $L=C$, in case $K=GF(q)$ we take $L=GF(q^{(k+1)!})$ (compare with [10]). Let $u_1u_2\cdots u_{k+1}=b\in K$. The $i$-th elementary symmetric function $\sigma_i$ in $u_1,\cdots,u_{k+1}$ is denoted by $x_i$, i.e.

$$\begin{cases} x_1=u_1+\cdots+u_{k+1}=\sigma_1(u_1,\cdots,u_{k+1})\\ x_2=u_1u_2+u_1u_3+\cdots+u_ku_{k+1}=\sigma_2(u_1,\cdots,u_{k+1})\\ \cdots\\ x_k=u_1\cdots u_k+u_1\cdots u_{k-1}u_{k+1}+\cdots+u_2u_3\cdots u_{k+1}=\sigma_k(u_1,\cdots,u_{k+1})\\ x_{k+1}=u_1u_2\cdots u_{k+1}=\sigma_{k+1}(u_1,\cdots,u_{k+1})=b. \end{cases} \tag{2.6}$$

We introduce a generalization of the Chebyshev polynomials of the first kind, using $\underset{\sim}{x}$ to denote the $k$-dimensional vector $(x_1,\cdots,x_k)$.

**Definition 2.1.** $P_{m,n}^{-1/2}(\underset{\sim}{x}\,;b)=\sum_{i=1}^{k+1}\sum_{\substack{j=1\\ j\neq i}}^{k+1}u_i^mu_j^{-n}$ for integers $m$, $n$ and a nonzero element $b$ in $K$.

These polynomials are $k$-dimensional generalizations of the polynomials $P_{m,n}^{-1/2}(x,y\,;1)$ introduced by Koornwinder [8] in case $K=C$ and also investigated in [3]. The special polynomials $P_{m,0}^{-1/2}(\underset{\sim}{x}\,;b)$, denoted by $kg_m(\underset{\sim}{x})$, have been introduced by Lidl and Wells [10] as $k$ times the $m$-th power sum of the roots of the polynomial

$$r(z)=z^{k+1}-x_1z^k+x_2z^{k-1}+\cdots+(-1)^kx_kz+(-1)^{k+1}b$$

over $K$. In case $K=C$ they were also introduced by Ricci [12]. using the notation from [8] or [3] for these polynomials, we have

$$P_{m,0}^{-1/2}(\underset{\sim}{x}\,;b)=k\sum_{i=1}^{k+1}u_i^m=kg_m(\underset{\sim}{x}\,;b) \tag{2.7}$$

and

$$P_{0,n}^{-1/2}(\underline{x}\,;b)=k\sum_{j=1}^{k+1}u_j^{-n}=kg_{-n}(\underline{x}\,;b). \tag{2.8}$$

Thus we can derive from Definition 2.1.

$$P_{m,n}^{-1/2}(\underline{x}\,;b)=\frac{1}{k^2}P_{m,0}^{-1/2}(\underline{x}\,;b)P_{0,n}^{-1/2}(\underline{x}\,;b)-\frac{1}{k}P_{m-n,0}^{-1/2}(\underline{x}\,;b). \tag{2.9}$$

In order to define generalized Chebyshev polynomials of the second kind we introduce the matrix $\mathfrak{U}_{m,n}$ of elements $u_1,\cdots,u_{k+1}$ in an extension $L$ of $K$, where $u_1\cdots u_{k+1}=b\in K$.

$$\mathfrak{U}_{m,n}=\begin{bmatrix} u_1^{m+k} & u_2^{m+k}\cdots u_{k+1}^{m+k} \\ u_1^{k-1} & u_2^{k-1}\cdots u_{k+1}^{k-1} \\ \cdots & \cdots\cdots\cdots\cdots \\ u_1 & u_2 \quad \cdots u_{k+1} \\ u_1^{-n} & u_2^{-n}\cdots u_{k+1}^{-n} \end{bmatrix} \quad \text{for } m,n\geq 0. \tag{2.10}$$

Let $\mathfrak{U}_{m,n}^{(-1)}$ denote the matrix which is obtained from $\mathfrak{U}_{m,n}$ by replacing $u_i$ by $u_i^{-1}$ for $i=1,2,\cdots,k+1$. Then we define polynomials over the field $K$ by

**Definition 2.2.** $P_{m,n}^{1/2}(\underline{x}\,;b)=(\det\mathfrak{U}_{m,n})(\det\mathfrak{U}_{0,0})^{-1}$

$P_{-m,-n}^{1/2}(\underline{x}\,;b)=(\det\mathfrak{U}_{m,n}^{(-1)})(\det\mathfrak{U}_{0,0}^{(-1)})^{-1}$

where the matrix $\mathfrak{U}_{m,n}$ is given by (2.10) and $\underline{x}=(x_1,\cdots,x_k)$,

$$x_i=\sigma_i(u_1,\cdots,u_{k+1}),\ 1\leq i\leq k+1, \quad\text{and}\quad x_{k+1}=b\in K.$$

Finally we generalize the polynomials $D_{m,n}^{-1/2}(x,y)$ introduced in [3] to the $k$-dimensional case.

**Definition 2.3.** The polynomials $D_{m_1,\cdots,m_k}^{-1/2}(\underline{x})$ in $k$ variables $(x_1,\cdots,x_k)=\underline{x}$ are given by the generating function

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}D_{m_1,\cdots,m_k}^{-1/2}(\underline{x})s_1^{m_1}\cdots s_k^{m_k}$$

$$=\frac{1-\left(1-\sum_{i=1}^{k}x_i^2\right)\left(\sum_{i=1}^{k}s_i^2\right)-\sum_{i=1}^{k}s_i^2x_i^2}{\left(1-\sum_{i=1}^{k}s_ix_i\right)^2+\left(1-\sum_{i=1}^{k}x_i^2\right)\left(\sum_{i=1}^{k}s_i^2\right)}.$$

As orthogonal polynomials over $\boldsymbol{R}$ on the hypershpere $\sum_{i=1}^{k}x_i^2=1$ with weight function $\left(1-\sum_{i=1}^{k}x_i^2\right)^{-1/2}$ these polynomials could be regarded as Chebyshev polynomials of the first kind in $k$ variables. Polynomials of the second kind can be defined by replacing the numerator of the generating function in Definition 2.3 by 1. Thus

**Definition 2.4.**

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}D_{m_1,\cdots,m_k}^{1/2}(\underline{x})s_1^{m_1}\cdots s_k^{m_k}=\frac{1}{(1-\sum s_ix_i)^2+(1-\sum x_i^2)(\sum s_i^2)}.$$

**3. Results.** We use the notation $\underline{x}=(x_1,\cdots,x_k)$, introduced in § 2. In Lemma 3.1 $\underline{x}'=(x_k,x_{k-1},\cdots,x_1)$.

**Lemma 3.1.** $P_{-m,0}^{-1/2}(\underline{x}\,;b)=P_{0,m}^{-1/2}(\underline{x}\,;b)=P_{m,0}^{-1/2}(b^{-1}\underline{x}'\,;b^{-1})$

$$P^{1/2}_{-m,0}(\underline{x}\,;b)=P^{1/2}_{0,m}(\underline{x}\,;b)=P^{1/2}_{m,0}(b^{-1}\underline{x}'\,;b^{-1}).$$

The restriction $b\neq 0$ in Definitions 2.1 and 2.2 are not crucial, because of

**Lemma 3.2.** $P^{-1/2}_{m,0}(\underline{x}\,;0)=P^{-1/2}_{m,0}(x_1,\cdots,x_{k-1}\,;x_k)$

$P^{1/2}_{m,0}(\underline{x}\,;0)=P^{1/2}_{m,0}(x_1,\cdots,x_{k-1}\,;x_k).$

From (2.9) and Lemma 3.1 we have,

**Lemma 3.3.** $P^{-1/2}_{m,n}(\underline{x}\,;b)=\dfrac{1}{k^2}P^{-1/2}_{m,0}(\underline{x}\,;b)P^{-1/2}_{-n,0}(\underline{x}\,;b)-\dfrac{1}{k}P^{-1/2}_{m-n,0}(\underline{x}\,;b)$

**Lemma 3.4.**
$$\frac{1}{k}\sum_{m=0}^{\infty}P^{-1/2}_{m,0}(\underline{x}\,;b)s^m=\frac{N_+}{D_+}$$
$$\frac{1}{k}\sum_{m=0}^{\infty}P^{-1/2}_{-m,0}(\underline{x}\,;b)t^m=\frac{N_-}{D_-}$$

where

$$N_+=\sum_{i=0}^{k}(k+1-i)(-1)^i x_i s^i \tag{3.1}$$

$$N_-=\sum_{i=0}^{k}(k+1-i)(-1)^i b^{-1}x_{k+1-i}t^i \tag{3.2}$$

$$D_+=\sum_{i=0}^{k+1}(-1)^i x_i s^i \tag{3.3}$$

$$D_-=\sum_{i=0}^{k+1}(-1)^i b^{-1}x_{k+1-i}t^i. \tag{3.4}$$

**Theorem 3.5** (Generating Function).
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P^{-1/2}_{m,n}(\underline{x}\,;b)s^m t^n=\frac{N_+N_--M}{D_+D_-}$$
*where* $(1-st)M=D_+N_-+D_-N_+-(k+1)D_+D_-$.

**Theorem 3.6** (Recurrence Relation).
$$P^{-1/2}_{m,n}=\sum_{i=1}^{k+1}(-1)^{i-1}x_i P^{-1/2}_{m-i,n}\qquad for\ m>k$$
$$P^{-1/2}_{m,n}=b^{-1}\sum_{i=1}^{k+1}(-1)^{i-1}x_{k+1-i}P^{-1/2}_{m,n-i}\qquad for\ n>k,$$
*where* $x_{k+1}=b,\ x_0=1$ *and the initial conditions are given by*
$$P^{-1/2}_{m,0}=\sum_{i=1}^{m}(-1)^{i-1}x_i P^{-1/2}_{m-i,0}+k(-1)^m(k+1-m)x_m\qquad for\ 0\le m\le k$$
$$P^{-1/2}_{-m,0}=\sum_{i=1}^{m}(-1)^{i-1}b^{-1}x_{k+1-i}P^{-1/2}_{-m+i,0}+k(-1)^m(k+1-m)b^{-1}x_{k+1-m}$$
$$for\ 0\le m\le k.$$
*and*
$$P^{-1/2}_{m,n}=\frac{1}{k^2}P^{-1/2}_{m,0}P^{-1/2}_{-n,0}-\frac{1}{k}P^{-1/2}_{m-n,0}.$$

In the special cases $n=0$, and $b=1$ these results have been obtained in [11]. We list some of the polynomials $P^{-1/2}_{m,n}(\underline{x}\,;b)$ of small degrees for $k=2$ and $k=3$.

| | $k=2$ | $k=3$ |
|---|---|---|
| $P_{00}^{-1/2}$ | $6$ | $12$ |
| $P_{01}^{-1/2}$ | $2b^{-1}y$ | $3b^{-1}z$ |
| $P_{02}^{-1/2}$ | $2b^{-2}(y^2-2bx)$ | $3b^{-2}(z^2-2by)$ |
| $P_{10}^{-1/2}$ | $2x$ | $3x$ |
| $P_{11}^{-1/2}$ | $b^{-1}(xy-3b)$ | $b^{-1}(xz-4b)$ |
| $P_{12}^{-1/2}$ | $b^{-2}(xy^2-2bx^2-by)$ | $b^{-2}(xz^2-2bxy-bz)$ |
| $P_{20}^{-1/2}$ | $2(x^2-2y)$ | $3(x^2-2y)$ |
| $P_{21}^{-1/2}$ | $b^{-1}(x^2y-2y^2-bx)$ | $b^{-1}(x^2z-2yz-bx)$ |
| $P_{22}^{-1/2}$ | $b^{-2}(x^2y^2-2y^3-2bx^3+4bxy-3b^2)$ | $b^{-2}(x^2z^2-bx^2yz-2yz^2+4by^2-4b^2)$ |

Generating functions and recurrence relations for the generalized Chebyshev polynomials $P^{1/2}_{m,n}$ of the second kind follow now. $D_+$ and $D_-$ are as in Lemma 3.4.

**Lemma 3.7.** $\sum_{m=0}^{\infty} P^{1/2}_{m,0}(\underline{x}\,;\,b)t^m=\dfrac{1}{D_+}$ and $\sum_{m=0}^{\infty} P^{1/2}_{-m,0}(\underline{x}\,;\,b)t^m=\dfrac{1}{D_-}$.

**Theorem 3.8** (Generating Function).

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P^{1/2}_{m,n}(\underline{x}\,;\,b)s^m t^n=\frac{1-st}{D_+D_-}.$$

**Theorem 3.9** (Recurrence Relation).

$$P^{1/2}_{m,n}=\sum_{i=1}^{\min(m,k+1)}(-1)^{i-1}x_iP^{1/2}_{m-i,n} \qquad \textit{for } m>1$$

$$P^{1/2}_{m,n}=\sum_{i=1}^{\min(n,k+1)}(-1)^{i-1}b^{-1}x_{k+1-i}P^{1/2}_{m,n-i} \qquad \textit{for } n>1$$

*where*

$$P^{1/2}_{0,0}=1,\quad P^{1/2}_{1,0}=x_1,\quad P^{1/2}_{0,1}=b^{-1}x_k \quad \textit{and} \quad P^{1/2}_{1,1}=b^{-1}x_1x_k-1.$$

*We can see that for $|m|>k$ the polynomials $P^{1/2}_{m,0}$ satisfy the same recurrence relation as $P^{-1/2}_{m,0}$.*

**Corollary 3.10.** $P^{1/2}_{m,0}=\sum_{i=1}^{m}(-1)^{i-1}x_iP^{1/2}_{m-i,0}$

$$P^{1/2}_{-m,0}=\sum_{i=1}^{m}(-1)^{i-1}b^{-1}x_{k+1-i}P^{1/2}_{-m+i,0}$$

*for* $0\leq m\leq k$.

**Lemma 3.11.** $P^{1/2}_{m,n}=P^{1/2}_{m,0}P^{1/2}_{-n,0}-P^{1/2}_{m-1,0}P^{1/2}_{-(n-0),0}$ *for* $m,n\neq 0$.

**Lemma 3.12.** $\dfrac{1}{k}P^{-1/2}_{i,0}(\underline{x}\,;\,b)=\sum_{i=0}^{\min(m,k)}(k+1-i)(-1)^i x_i P^{1/2}_{m-i}(\underline{x}\,;\,b)$

*for* $m\geq 0$.

We list some of the polynomials $P^{1/2}_{m,n}$ of small degrees for $k=2$ and $k=3$.

| | $k=2$ | $k=3$ |
|---|---|---|
| $P^{1/2}_{0,0}$ | $1$ | $1$ |
| $P^{1/2}_{0,1}$ | $b^{-1}y$ | $b^{-1}z$ |
| $P^{1/2}_{0,2}$ | $b^{-2}(y^2-bx)$ | $b^{-2}(z^2-by)$ |

| | $k=2$ | $k=3$ |
|---|---|---|
| $P^{1/2}_{1,0}$ | $x$ | $x$ |
| $P^{1/2}_{1,1}$ | $b^{-1}(xy-b)$ | $b^{-1}(xz-b)$ |
| $P^{1/2}_{1,2}$ | $b^{-2}(xy^2-bx^2-by)$ | $b^{-2}(xz^2-bxy-bz)$ |
| $P^{1/2}_{2,0}$ | $x^2-y$ | $x^2-y$ |
| $P^{1/2}_{2,1}$ | $b^{-1}(x^2y-y^2-bx)$ | $b^{-1}(x^2z-yz-bx)$ |
| $P^{1/2}_{2,2}$ | $b^{-2}(x^2y^2-bx^3-y^3)$ | $b^2(x^2z^2-bx^2y-yz^2+by^2-bxz)$ |

Finally we have the following relationship between the polynomials introduced in Definitions (2.3), (2.4) and (2.1), (2.2).

**Theorem 3.13.** $D^{-1/2}_{0,\dots,m_i,\dots,0}(\underline{x})=\dfrac{1}{k}P^{-1/2}_{m_i,0}\left(2x_i\,;1-\sum\limits_{\substack{j=1\\j\neq i}}^{k} x_j^2\right)\qquad m_i\neq 0,$

$$D^{1/2}_{0,\dots,m_i,\dots,0}(\underline{x})=P^{1/2}_{m_i,0}\left(2x_i\,;1-\sum_{\substack{j=1\\j\neq i}}^{k} x_j^2\right)\qquad m_i\neq 0.$$