77. *Cross Ratios as Moduli of del Pezzo Surfaces of Degree One*

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1. The projective plane with eight points on it blown up is called a del Pezzo surface of degree 1 provided the points are in general position, Manin [2]. For any such surface the linear system of anti-canonical divisors has exactly one fixed point, and if one further blows it up, then one obtains a rational elliptic surface with only irreducible fibers. According to Kodaira [1], the global sections of such an elliptic surface are exactly the exceptional curves of the first kind. Suppose conversely that there is given a rational elliptic surface with only irreducible fibers. Then, if one blows down one of the global sections, one obtains a del Pezzo surface of degree 1, and the isomorphism class of this del Pezzo surface does not depend on the choice of the global section to be blown down. We therefore prefer to study rational elliptic surfaces rather than del Pezzo surface of degree 1; for, much is known about the former.

Now let S be a rational elliptic surface with only irreducible fibers. Then the Néron-Severi group $N(S)$ (=the second homology group) of S is generated by the classes of global sections and the canonical class K; Shioda [4]. (The fibers belong to $-K$; i.e., they are the anti-canonical divisors of S.) Since $K^2=0$, the orthogonal complement $K^{\perp}=\{D \in N(S)\,;\, D\cdot K=0\}$ contains the multiples $\{K\}$ of K. We set now

$$\Gamma(S)=K^{\perp}/\{K\}$$

and call it the module associated with S. $\Gamma(S)$ is free $\boldsymbol{Z}$-module of rank 8 and the intersection form induces a negative definite, even, integral quadratic form of determinant 1 on $\Gamma(S)$. Therefore there are isometries between $\Gamma(S)$ and the module Γ of weights of E_8 with respect to this induced form and the Killing form. The surface S endowed with an isometry α of Γ onto $\Gamma(S)$ is called a *marked rational elliptic surface* or simply an *MRE surface*, and is denoted by (S,α). For an MRE surface, the fibers are assumed to be irreducible throughout this note. Two MRE surfaces (S,α), (S',α') are called *isomorphic* if there is a biregular morphism β of S onto S' such that $\beta_*\alpha=\alpha'$ where β_* denotes the isometry of $\Gamma(S)$ onto $\Gamma(S')$ induced by β. The purpose of this note is to explain briefly how to describe the set $\mathcal{M}$ of isomorphism classes of MRE surfaces as an algebraic variety.

2. Instead of studying $\mathcal{M}$ directly, we first observe the MRE surfaces endowed with some more additional structure: $(S, \alpha; C)$ is called *an overmarked rational elliptic surface* or an *ORE surface* for short if (S, α) is an MRE surface and C is a singular fiber of the elliptic surface S. Since C is assumed to be irreducible, C must be a rational curve either with one ordinary double point or with one cusp i.e. C is either of type I_1 or of type II in the notation of [1]. The condition under which two ORE surfaces are said to be isomorphic might be regarded as clear. We denote by $\hat{\mathcal{M}}$ the set of isomorphism classes of ORE surfaces. Now let $(S, \alpha; C)$ be an ORE surface and assume that C is of type I_1. Then there is a biregular morphism ι of $C^* = C - \mathrm{Sing}(C)$ onto the algebraic group $k^* = k - \{0\}$ where k is the ground field assumed to be algebraically closed. ($\mathrm{Sing}(C)$ stands for the set of singular points of C.)

Lemma 1. *The assumption being as above, there is one and only one element h of $\mathrm{Hom}_Z(\Gamma(S), k^*)$ such that, for any two global sections σ, σ' of S,*

$$h([\sigma'] - [\sigma] + \{K\}) = \iota(C \cap \sigma')/\iota(C \cap \sigma).$$

We have denoted by $[\sigma], [\sigma']$ the homology classes of σ, σ' respectively. Note that $[\sigma'] - [\sigma]$ belongs to $K^{\perp}$. In view of this lemma, we can now assign to $(S, \alpha; C)$ a definite element of $T/\{\pm 1\}$ where T denotes the maximal torus $\mathrm{Hom}_Z(\Gamma, k^*)$ of E_8 and $\{\pm 1\}$ the center of the Weyl group $W(E_8)$. This element is just to be the class in $T/\{\pm 1\}$ of the image of h in Lemma 1 under the isomorphism α^* of $\mathrm{Hom}_Z(\Gamma(S), k^*)$ to $T = \mathrm{Hom}_Z(\Gamma, k^*)$ induced by α. We denote it by $\rho(S, \alpha; C)$; i.e. $\rho(S, \alpha; C) = [\alpha^*(h)]$.

Next we want to naturally assign a similar object to each ORE surface $(S, \alpha; C)$ such that C is of type II. We first denote by $\bar{T} \to T$ the blowing up with center $1 \in T$ and by E the associated exceptional divisor. Note that there is a natural projection of the Lie algebra $\mathcal{T} = \mathrm{Hom}(\Gamma, k)$ of T minus zero onto E. Assume now that C is of type II and observe that there is a biregular morphism ι of $C^* = C - \mathrm{Sing}(C)$ onto the additive group k.

Lemma 2. *The notation and assumption being as just above, there is a unique element h of $\mathrm{Hom}(\Gamma(S), k)$ such that, for any two global sections σ and σ',*

$$h([\sigma'] - [\sigma] + \{K\}) = \iota(C \cap \sigma') - \iota(C \cap \sigma).$$

Obviously $\alpha^*(h)$ is a non-zero element of $\mathcal{T}$, so the canonical projection maps it to an element of $E \cong E/\{\pm 1\}$, which we denote also by $\rho(S, \alpha; C)$.

Proposition 1. *The element $\rho(S, \alpha; C)$ depends only on the isomorphism class of the ORE surface $(S, \alpha; C)$. The assignment*

$(S,\alpha;C)\to\rho(S,\alpha;C)$ *induces a bijection of* $\hat{\mathcal{M}}$ *onto the open set* $(\bar{T}-\bar{\Delta})/\{\pm 1\}$ *of* $\bar{T}/\{\pm 1\}$ *where* $\bar{\Delta}$ *denotes the strict transform of the union* Δ *of the fixed point sets in* T *of the* 120 *reflexions of* $W(E_8)$.

It should be remarked here that the mapping $\rho:\hat{\mathcal{M}}\rightrightarrows(\bar{T}-\bar{\Delta})/\{\pm 1\}$ is just an algebraization of the idea of Looijenga of constructing moduli spaces of marked del Pezzo surfaces, which is explained in the appendix of [3]. But in the case of degree 1 it exceptionally gives us not the moduli $\mathcal{M}$ but $\hat{\mathcal{M}}$ because the Weyl group does not transitively permute the singular anti-canonical divisors. This note in fact grew out of the attempt to overcome this difficulty of the method.

3. In this final section we will show that the cross ratios such as introduced in [3] enables us to obtain a finite morphism of $(\bar{T}-\bar{\Delta})/\{\pm 1\}$ $(\simeq\hat{\mathcal{M}})$ onto a certain algebraic variety which is to parametrize $\mathcal{M}$. To explain what they are, we regard all roots of E_8 as k^*-characters on T. Let $\chi_1,\chi_2,\chi_3,\chi_4;\chi'_1,\chi'_2,\chi'_3,\chi'_4$ be eight roots of E_8 such that, for $i\neq j$, χ_i,χ_j (resp. χ'_i,χ'_j) are orthogonal (with respect to the Killing form), that $\chi_i\chi'_i$ $(i=1,2,3,4)$, χ_i/χ'_j $(i\neq j)$ are roots, and that $\chi_i\chi_j=\chi'_k\chi'_l$ provided $\{i,j,k,l\}=\{1,2,3,4\}$. Then the rational functions such as of the form

$$\frac{(\chi_1-1)(\chi_2-1)(\chi_3-1)(\chi_4-1)}{(\chi'_1-1)(\chi'_2-1)(\chi'_3-1)(\chi'_4-1)}$$

are called the cross ratios of type D_4 associated with E_8. (Note that χ_i's and χ'_i's generate a root system of type D_4; $\chi_1,\chi_2,\chi_3,(\chi'_4)^{-1}$ for example form a fundamental system of roots of D_4.) They can all be regarded as regular functions on $(\bar{T}-\bar{\Delta})/\{\pm 1\}$. If z is one of the cross ratios of this type, then $z^{\pm 1}$, $(1-z)^{\pm 1}$, $\{(z-1)/z\}^{\pm 1}$ are all among them. Let now N be the number of the cross ratios $(N=3150\times 6)$ and suppose that they are indexed: r_i $(i=1,2,\cdots,N)$. We then obtain morphism r given by

$$r:(\bar{T}-\bar{\Delta})/\{\pm 1\}\ni p\mapsto(r_1(p),r_2(p),\cdots,r_N(p))\in k^N.$$

Proposition 2. *The image* $r((\bar{T}-\bar{\Delta})/\{\pm 1\})$ *is a* 8*-dimensional closed submanifold of* k^N*, and there exists a unique bijection* $\check{\rho}$ *of* $\mathcal{M}$ *onto the image that makes the following diagram commutative*:

$$\begin{tikzcd}
\hat{\mathcal{M}} \arrow[r, "\rho"] \arrow[d, "\pi"'] & (\bar{T}-\bar{\Delta})/\{\pm 1\} \arrow[d, "r"] \\
\mathcal{M} \arrow[r, "\check{\rho}"] & r((\bar{T}-\bar{\Delta})/\{\pm 1\})
\end{tikzcd}$$

where π *is the canonical projection* $(S,\alpha;C)\mapsto(S,\alpha)$.

The finite morphism r is of degree 12 since a rational elliptic surface has exactly 12 singular fibers of type I_1 in general, and the ramification of r occurs just along the divisor $(E-\bar{\Delta})/\{\pm 1\}$: Let $p\in r((\bar{T}-\bar{\Delta})/\{\pm 1\})$, and let s be the number of points in $r^{-1}(p)$ $\cap(E-\bar{\Delta})/\{\pm 1\}$ and t the number of the other points in $r^{-1}(p)$. Then $2s+t=12$ by Kodaira [1].

Before closing this note, we should explain why the cross ratios $r_i(\rho(S, \alpha; C))$ $(i=1, 2, \cdots, N)$ do not depend on the specification of the singular fiber C so that the composition $r \circ \rho$ is factored by π. The maximal number of non-intersecting global sections of S is nine provided S has only irreducible fibers. Let $\sigma_1, \sigma_2, \cdots, \sigma_9$ be such nine global sections and $\varphi: S \rightarrow P_2$ the blowing down of them. Then the images of $\sigma_1, \sigma_2, \cdots, \sigma_9$ under φ are points $p_1, p_2, \cdots, p_9$ on P_2 respectively. Choose arbitrary five points out of these nine, say, $p_1, p_2, \cdots, p_5$. and consider four lines $\overline{p_1p_2}, \overline{p_1p_3}, \cdots, \overline{p_1p_5}$ passing through p_1. Since the totality of lines passing through one point is naturally regarded as the projective line, the four lines are regarded as four points on P_1. The cross ratio that these points define is one of the cross ratios considered above. We thus obtain $9 \times \binom{8}{4} = 630$ systems of the cross ratios by permuting the p_i's. They are obviously independent of the singular cubic curves that pass through the nine points, which are exactly the singular fibers of the elliptic surface S. The other cross ratios can also be obtained by replacing the σ_i's by other suitable nine non-intersecting global sections. Thus we have explained the geometric meaning of the cross ratios of type D_4.

We finally remark the followings: The Weyl group $W(E_8)$ operates naturally on $\mathcal{M}, \hat{\mathcal{M}}, (\bar{T}-\bar{\Delta})/\{\pm 1\}, r((\bar{T}-\bar{\Delta})/\{\pm 1\})$ and the mappings $\pi, \rho, \check{\rho}, r$ are obviously $W(E_8)$-equivariant. The quotient of $\mathcal{M}$ by this operation is just the coarse moduli space of del Pezzo surfaces of degree one, the structure of which can in principle be seen by the explicit description $\mathcal{M} \simeq r((\bar{T}-\bar{\Delta})/\{\pm 1\})$.

References

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