92. On Semipositive Line Bundles

By Takao FUJITA
University of Tokyo and University of California*)

(Communicated by Kunihiko KODAIRA, M. J. A., Oct. 13, 1980)

Unlike the positivity of line bundles, in which case several different definitions turn to be equivalent to each other, there are a couple of really different notions of semipositivity. Here we want to clarify the situation as well as possible now. Details and proofs will be published elsewhere.

For the sake of simplicity we work in the category of projective K-schemes, where K is an algebraically closed field. K is assumed to be the complex number field in the statements indicated by $/C$. A variety means an irreducible, reduced projective K-scheme. Tensor products of line bundles are denoted additively, and are regarded as rational equivalence classes in Chow ring.

§ 1. Definitions and interrelations.

(1.1) **Definition.** Let S be a projective K-scheme and let L be a line bundle on S. Then L is said to be

a) *semiample*, if $\mathcal{O}_S[mL]$ is generated by global sections for some $m>0$;

b) *cohomologically semipositive* (abbreviation: *c-semipositive*), if, for any coherent sheaf $\mathcal{F}$ on S and for any very ample line bundle H on S, there is an integer a such that $H^p(\mathcal{F}[tL+sH])=0$ for any $p>0$, $t\geqq 0$, $s>a$;

c) *approximately ample*, if there is a line bundle F on S and a positive integer m such that $\mathcal{O}_S[F+tmL]$ is generated by global sections for any $t\geqq 0$;

d) *numerically semipositive* (abbr: *n-semipositive*), if $LC\geqq 0$ for any curve C in S;

e) *universally effective*, if, for any subvariety V of S, there exists a positive integer m such that $H^0(V', mL_{V'})\neq 0$, where V' is the normalization of V;

f) *geometrically semipositive* (abbr: *g-semipositive*), if S is a complex manifold and $c_1(L)$ is represented by a closed Hermitian (1, 1)-form which is everywhere positive semidefinite.

(1.2) **Theorem.** *All the above notions* a)–f) *satisfy the following axioms.*

---

*) This article was completed when the author was a Miller Fellow at Berkeley.

(A1) *Any positive (=ample) line bundle is semipositive.*

(A2) *L is semipositive if and only if mL is so for some positive integer m.*

(A3) *$L+H$ is positive if L is semipositive and if H is positive.*

(A4) *For any morphism $f: T \to S$, f^* $L$is semipositive if L is so on S.*

(1.3) **Theorem.** *Numerical semipositivity is the weakest one among all the semipositivities satisfying the axioms* (A1)–(A4). *Semi-ampleness is the strongest among them.*

(1.4) **Theorem.** *The following implications are valid*: a)⇨e)⇨b)⇨c)⇨d). *Moreover, if $K=\boldsymbol{C}$,* a)⇨f)⇨b) *and* d)⇨b) *are also true. In particular,* b), c) *and* d) *are equivalent to each other over* $\boldsymbol{C}$.

Remark. Our proof of d)⇨b)/$\boldsymbol{C}$ depends on the vanishing theorem of Kodaira and the theory of resolution of singularities by Hironaka.

(1.5) Considering the case in which S is an elliptic curve for example, we see that there are two types of semipositivities. b), c), d) and f) are of the weaker one, and a) and e) are of the stronger one. Those of the latter type are not stable under small deformation, while those of the former type have the tendency to be so.

Question. Are the properties b), c), d) and f) equivalent to each other? Are the properties a) and e) equivalent to each other?

§ 2. Properties of semipositive line bundles.

(2.1) **Theorem.** *Let $L_1, \cdots, L_r$ be numerically semipositive line bundles on a variety V with* $\dim V=r$. *Then* $L_1 \cdots L_r\{V\} \geqq 0$.

This result is essentially due to Kleiman [3]. Our n-semipositivity is identical with his pseudo-ampleness. (See also [2].)

(2.2) **Corollary.** *Let L be a n-semipositive line bundle on V. Then there is an integer σ with $0 \leqq \sigma \leqq r$ such that, for any ample line bundle H on V, $L^a H^{r-a}>0$ for any $a \leqq \sigma$ and $L^a H^{r-a}=0$ for any $a>\sigma$.*

(2.3) **Definition.** The above integer σ will be called the *index of positivity* of L and is denoted by $\sigma(L)$.

Remark. If L is semiample, then $\sigma(L)=\kappa(L):=\mathrm{Max}_{m>0} \cdot(\dim \rho_{|mL|}(V))$ where $\rho_{|mL|}$ is the rational mapping defined by $|mL|$. However, for general n-semipositive line bundles, we have only the inequality $\sigma(L) \geqq \kappa(L)$.

(2.4) **Theorem.** *Let L be a geometrically semipositive line bundle on a Kaehler manifold M. Then $H^p(M, \mathcal{O}_M[-L])=0$ for any $p<\sigma(L)$.*

Here, we modify the definition of $\sigma(L)$ suitably when M is not projective.

(2.5) **Theorem.** *Let L be a cohomologically semipositive line* *bundle on a variety V with* dim $V=n$. *Suppose that* $L^n\{V\}>0$. *Then* $\kappa(L, V)=n$.

(2.6) **Corollary/*C*.** *Let L be an n-semipositive line bundle on a variety V with* dim $V=n$. *Then,* $\kappa(L, V)=n$ *if and only if* $\sigma(L)=n$, *or equivalently,* $L^n>0$.

(2.7) **Theorem.** *Let L be a semiample line bundle on a K-scheme S. Then the graded K-algebra* $G(S, L)=\bigoplus_{t\geq 0} H^0(S, tL)$ *is finitely generated. Moreover, for any coherent sheaf* $\mathcal{F}$ *on S,* $\bigoplus_{t\geq 0} H^0(S, \mathcal{F}[tL])$ *is a finitely generated* $G(S, L)$*-module.*

Remark. The above conclusion does not hold to be true for semipositivities of weaker type. Besides (2.7), semiample line bundles have many nice properties. However, usually, it is not easy to verify a given line bundle to be semiample. If e)⇨a) is true, this would be a convenient criterion since the universal effectivity is much easier to show. Here we present the following

(2.8) **Theorem.** *Let L be a line bundle on a K-scheme S. Suppose that the restriction of L to the base locus* $Bs|L|$ *of L is ample. Then L is semiample.*

In particular, L is semiample if $Bs|L|$ is a finite set. Thus we generalize a well known result of Zariski (cf. [4], p. 579).

(2.9) **Definition.** A line bundle L on a variety V is said to be *almost base-point free* (in the stronger sense) if, for any $x \in V$, there exists a constant $N>0$ such that $\mathrm{Min}_{D\in|mL|}\, \mu_x(D)<N$ for any $m\gg 0$, where $\mu_x(D)$ denotes the multiplicity of the divisor D at x.

Remark. The above definition is slightly different from that of Goodman [1]. Anyway, he proved that an almost base-point free line bundle is n-semipositive.

(2.10) **Theorem.** *Let L be an approximately ample line bundle on a variety V such that* dim $V=\kappa(L, V)$. *Then L is almost basepoint free in the strong sense.*

(2.11) **Corollary/*C*.** *Let L be an n-semipositive line bundle on a variety V with* $L^n>0$, *where* $n=$ dim V. *Then L is almost basepoint free in the strong sense. Hence the graded algebra* $G(V, L)$ *is finitely generated if and only if L is semiample.*

§3. **Summary.** The notions of n-semipositivity and semiampleness are surely very important and useful (cf. (1.3), (2.1), (2.7) and (2.11)).

The c-semipositivity and approximate ampleness play cruicial roles in many technical arguments (cf. (2.5), (2.10)). In characteristic zero cases, they can be viewed as important properties of n-semipositive line bundles (cf. (1.4)).

The importance of g-semipositivity lies in the vanishing theorem (2.4).

The universal effectiveness is one of the most useful criterion for c-semipositivity (cf. (1.4)). If it implies semiampleness, this would be very useful.

References

[1] J. E. Goodman: Affine open subsets of algebraic varieties and ample divisors. Ann. of Math., **89**, 160–183 (1969).

[2] R. Hartshorne: Ample subvarieties of Algebraic Varieties. Lect. Notes in Math., vol. 156, Springer (1970).

[3] S. Kleiman: Towards a numerical theory of ampleness. Ann. of Math., **84**, 293–344 (1966).

[4] O. Zariski: The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. Ibid., **76**, 560–615 (1962).