84. *Some Examples of Analytic Functionals with Carrier at the Infinity*

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In this note we propose some examples of analytic functionals with carrier at the infinity. In particular, we will give an example of a Fourier hyperfunction with support at the infinity.

We confine ourselves to the one dimensional case and follow the notations in Morimoto [2] and Morimoto-Yoshino [3]. Let $L=A+iK$, $i=\sqrt{-1}$, $A=[a,\infty)$, $K=[-k,k]$ and $k'\in\boldsymbol{R}$. We denote by $Q_b(L;k')$ the space of all continuous functions f on L holomorphic in the interior of L which satisfy the following condition:

(1) $$\sup\{|f(\zeta)|\exp(k'\xi);\zeta=\xi+i\eta\in L\}<\infty.$$

Taking the inductive limit following the restriction mappings as $\varepsilon\downarrow 0$ and $\varepsilon'\downarrow 0$, we define the fundamental space

(2) $$Q(L;k')=\underset{\varepsilon\downarrow 0\ \ \varepsilon'\downarrow 0}{\lim\ \mathrm{ind}}\ Q_b(L_\varepsilon;k'+\varepsilon'),$$

where $L_\varepsilon=[a-\varepsilon,\infty)+i[-k-\varepsilon,k+\varepsilon]$. A continuous linear functional S on the space $Q(L;k')$ is, by definition, an analytic functional with carrier in L and of exponential type k'. $Q'(L;k')$ will denote the dual space of $Q(L;k')$. An analytic functional S is said to be with carrier in $\infty+iK$ if $S\in Q'([a,\infty)+iK;k')$ for every $a>0$.

We recall three transformations of analytic functionals:

1) The Cauchy transformation of $S\in Q'(L;k')$ is defined by the following formula:

(3) $$\check{S}(\tau)=\frac{-1}{2\pi i}\left\langle S_\zeta,\frac{\exp(-(\tau-\zeta)^2)}{\tau-\zeta}\right\rangle.$$

It is known that $\check{S}(\tau)$ is a holomorphic function on $\boldsymbol{C}\backslash L$, satisfying, for any positive numbers ε, r and ε' with $0<\varepsilon<r$,

(4) $$\sup\{|\check{S}(\tau)|\exp(-(k'+\varepsilon')s);\tau=s+it\in L_r\backslash L_\varepsilon\}<\infty$$

and that we have the inversion formula

(5) $$\langle S,f\rangle=-\int_{\partial L_\varepsilon}\check{S}(\tau)f(\tau)d\tau$$

for every $f\in Q(L;k')$, where $\varepsilon>0$ is sufficiently small (Theorems 3.2 and 3.3 in Morimoto [2]).

2) The Fourier-Borel transformation $\tilde{S}$ of $S\in Q'(L;k')$ is defined by

(6) $$\tilde{S}(z)=\langle S_\zeta,\exp(z\zeta)\rangle.$$

It is known the Fourier-Borel transformation $\mathcal{F}:S\mapsto\tilde{S}$ establishes a topological linear isomorphism of $Q'(L;k')$ onto $\mathrm{Exp}((-\infty,-k')+i\boldsymbol{R};L)$, the space of all holomorphic functions F on the left half plane $(-\infty,-k')+i\boldsymbol{R}$ which satisfy the condition: For any $\varepsilon>0$ and $\varepsilon'>0$, there exists a constant $C\geqq 0$ such that

$$|F(z)|\leqq C\exp((a-\varepsilon)x+(k+\varepsilon)|y|) \tag{7}$$

for $z=x+iy\in(-\infty,-k'-\varepsilon')+i\boldsymbol{R}$ (Theorem 5.1 in Morimoto [2]).

3) Suppose $0\leqq k<\pi$ and $k'<1$. Then the Avanissian-Gay transformation $G_S(w)$ of $S\in Q'(L;k')$ can be defined as follows:

$$G_S(w)=\langle S_\zeta,(1-we^\zeta)^{-1}\rangle. \tag{8}$$

The Avanissian-Gay transformation $G:S\mapsto G_S$ establishes a topological linear isomorphism of $Q'(L;k')$ onto $\mathcal{O}_0(\boldsymbol{C}\backslash\exp(-L);k')$, the space of all holomorphic functions on $\boldsymbol{C}\backslash\exp(-L)$, which vanish at $w=\infty$ and satisfy the following condition: For any ε with $0<\varepsilon<\pi-k$ and any ε' with $0<\varepsilon'<1-k'$, there exists a constant $C\geqq 0$ such that

$$|G_S(w)|\leqq C|w|^{-k'-\varepsilon'} \tag{9}$$

for $w\in\boldsymbol{C}\backslash(0)$ with $k+\varepsilon\leqq\arg w\leqq 2\pi-k-\varepsilon$ (Theorem 6 in Morimoto-Yoshino [3]). We have the inversion formula

$$\langle S,f\rangle=\frac{1}{2\pi i}\int_{\partial L_\varepsilon}G_S(e^{-\zeta})f(\zeta)d\zeta \tag{10}$$

for $f\in Q(L;k')$, where $\varepsilon>0$ is sufficiently small. The Laurent expansion of $G_S(w)$ can be given by the Fourier-Borel transformation $\tilde{S}$ of S as follows:

$$G_S(w)=-\sum_{n=1}^{\infty}\tilde{S}(-n)w^{-n}\qquad\text{for } |w|>e^{-a}. \tag{11}$$

Example 1. Suppose $L=[a,\infty)+i[-\pi/2,\pi/2]$ and $k'\in\boldsymbol{R}$. Let us define an analytic functional T by the formula

$$T:\varphi\mapsto\frac{1}{2\pi i}\int_{\partial L_\varepsilon}\varphi(\zeta)\exp(e^\zeta)d\zeta, \tag{12}$$

where $\varepsilon>0$ is sufficiently small. It is clear that $T\in Q'(L;k')$ for any a and any k'. Let us calculate the Fourier-Borel transformation $\tilde{T}$ of the analytic functional T. Putting $u=-e^\zeta$, we have

$$\begin{aligned}\tilde{T}(z)&=\frac{1}{2\pi i}\int_{\partial L_\varepsilon}\exp(z\zeta)\exp(e^\zeta)d\zeta\\&=\frac{1}{2\pi i}\int_{\partial L_{\pi/2}}\exp(z\zeta)\exp(e^\zeta)d\zeta\\&=\frac{1}{2\pi i}\int_\infty^{(0+)}(-u)^{z-1}e^{-u}du\\&=-\Gamma(1-z)^{-1},\end{aligned} \tag{13}$$

where the last equality results from the Hankel integral formula for the Γ function.

By (5), we have the following estimate of the Γ function: For any $R\in\boldsymbol{R}$, $M>0$ and $\varepsilon>0$, there exists $C\geqq 0$ such that

$$|\Gamma(z)|^{-1} \leqq C \exp\left(-Mx + \left(\frac{\pi}{2} + \varepsilon\right)|y|\right) \qquad \text{for } x = \operatorname{Re} z \geqq R.$$

If we put $G(w) = \exp(w^{-1}) - 1$, then it is clear

$$G(w) \in \mathcal{O}_0(\boldsymbol{C} \backslash \exp((-\infty, -a] + i[-\pi/2, \pi/2]) ; k')$$

for every $a \in \boldsymbol{R}$ and $k' < 1$ and that we have

$$\langle T, \varphi \rangle = \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \varphi(\zeta) \exp(e^\zeta) d\zeta = \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \varphi(\zeta) G(e^{-\zeta}) d\zeta.$$

Therefore, the Avanissian-Gay transformation $G_T(w)$ of T is the function $G(w)$:

(14) $$G_T(w) = \exp(w^{-1}) - 1.$$

The formula (11) reduces in this case to the following well known Taylor expansion:

$$\exp(w^{-1}) - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} w^{-n} \qquad \text{for } |w| > 0.$$

The analytic functional T defined above is an analytic functional with carrier in $\infty + i[-\pi/2, \pi/2]$. Considering the function $\exp(e^{M\zeta})$, for every $M > 0$ we can construct similarly an analytic functional with carrier in $\infty + i[-\pi/2M, \pi/2M]$.

Example 2. Let L and k' be as in Example 1. Suppose $\lambda > 0$ and define an analytic functional T_λ by

(15) $$T_\lambda : \varphi \mapsto \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \varphi(\zeta) \exp(\lambda \sinh \zeta) d\zeta.$$

It is clear that $T \in Q'(L; k')$ for any a and k'. Let us calculate the Fourier-Borel transformation of the functional T_λ. Putting $u = -e^\zeta$, we have

(16) $$\begin{aligned} \tilde{T}_\lambda(z) &= \frac{1}{2\pi i} \int_{\partial L_\varepsilon} \exp(z\zeta) \exp(\lambda \sinh \zeta) d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial L_{\pi/2}} \exp(z\zeta) \exp(\lambda \sinh \zeta) d\zeta \\ &= \frac{1}{2\pi i} \int_{\infty}^{(0+)} (-u)^{z-1} \exp((\lambda/2)(u^{-1} - u)) du \\ &= -J_{-z}(\lambda), \end{aligned}$$

where the last equality results from the Sonine integral formula for the Bessel functions.

By (5), we have the following estimate of the Bessel functions: For any $R \in \boldsymbol{R}$, $M > 0$ and $\varepsilon > 0$, there exists $C \geqq 0$ such that

$$|J_z(\lambda)| \leqq C \exp\left(-Mx + \left(\frac{\pi}{2} + \varepsilon\right)|y|\right) \qquad \text{for } x = \operatorname{Re} z \geqq R.$$

If we denote by $G_{T_\lambda}(w)$ the Avanissian-Gay transformation of the functional T_λ, we have by (11)

(17) $$G_{T_\lambda}(w) = \sum_{n=1}^{\infty} J_n(\lambda) w^{-n} = \exp((\lambda/2)(w^{-1} - w)) - \sum_{m=0}^{\infty} J_{-m}(\lambda) w^m$$

for $|w|>0$, where the second equality results from the generating formula of the Bessel functions.

Example 3. Put $H_{M,\pi}=\{\zeta=\xi+i\eta\in \boldsymbol{C};\ \xi\geqq M, |2\xi\eta|\leqq\pi\}$. Suppose $\varphi\in Q([a,\infty);k')$ and let $A_\varepsilon=[a-\varepsilon,\infty)+i[-\varepsilon,\varepsilon]$ be a definition domain of the function φ. If we choose a number M sufficiently large, we can assume the set $H_{M,\pi}$ is strictly contained in A_ε. Therefore we can define an analytic functional T as follows:

$$T:\varphi\mapsto\frac{1}{2\pi i}\int_{\partial H_{M,\pi}}\varphi(\zeta)\exp(\exp(\zeta^2))d\zeta, \tag{18}$$

where M is a sufficiently large number. By the Cauchy integral theorem, we can see the integral (18) is independent of such M. T is an analytic functional belonging to $Q'([a,\infty),k')$ for any $a\in\boldsymbol{R}$ and $k'\in\boldsymbol{R}$. For any $t>0$, the function $\zeta\exp(-t\zeta^2)$ belongs to $Q(\boldsymbol{R};k')$ for every $k'\in\boldsymbol{R}$. We have

$$\begin{aligned}\langle T_\zeta,\zeta\exp(-t\zeta^2)\rangle&=\frac{1}{2\pi i}\int_{\partial H_{M,\pi}}\zeta\exp(-t\zeta^2)\exp(\exp(\zeta^2))d\zeta\\&=\frac{1}{2\pi i}\frac{1}{2}\int_{\partial L_{\pi/2}}\exp(-t\tau)\exp(e^\zeta)d\zeta\\&=-\frac{1}{2}\Gamma(1+t)^{-1}\neq0,\end{aligned}$$

where L is given in Example 1 and we used (13). Therefore the analytic functional T does not vanish identically. If we consider T as a Fourier hyperfunction (Kawai [1] and Sato [4]), T is a Fourier hyperfunction whose support is concentrated to the infinity: $\operatorname{supp} T=\{+\infty\}$.

Let us define an entire function F as follows:

$$\begin{aligned}F(\tau)&=-2\pi i\check{T}(\tau)\\&=\frac{1}{2\pi i}\int_{\partial H_{M,\pi}}\frac{\exp(-(\tau-\zeta)^2)}{\tau-\zeta}\exp(\exp\zeta^2)d\zeta.\end{aligned}$$

Then the function F satisfies the condition (4) with $L=A=[a,\infty)$ and we have by (5)

$$\begin{aligned}\langle T,\varphi\rangle&=\frac{1}{2\pi i}\int_{\partial H_{M,\pi}}\varphi(\zeta)\exp(\exp\zeta^2)d\zeta\\&=\frac{1}{2\pi i}\int_{\partial L_\varepsilon}\varphi(\zeta)F(\zeta)d\zeta\end{aligned}$$

for $\varphi\in Q([a,\infty);k')$, where M is a sufficiently large number and ε is a sufficiently small positive number.

The explicit form of the function $F(\tau)$ is not known to us. We cannot calculate explicitly the Fourier-Borel transformation and the Avanissian-Gay transformation of the functional T. Similarly, the function $\exp(\lambda\sinh\zeta^2)$ gives another Fourier hyperfunction with support at the infinity.

Consider now the n-dimensional case. Let us define a functional

$T \in Q'(\boldsymbol{R}^n; 0)$ as follows:

$$\langle T, \varphi\rangle = \left(\frac{1}{2\pi i}\right)^n \int\cdots\int_{\partial H_{M,\pi}\times\cdots\times\partial H_{M,\pi}} \varphi(\zeta_1, \cdots, \zeta_n)$$
$$\times \exp(\exp\zeta_1^2 + \exp\zeta_2^2 + \cdots + \exp\zeta_n^2) d\zeta_1 d\zeta_2 \cdots d\zeta_n$$

for $\varphi \in Q(\boldsymbol{R}^n; 0) = \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^n)$, where $\boldsymbol{D}^n$ is the radial compactification of $\boldsymbol{R}^n$ (Kawai [1]). Then T is a Fourier hyperfunction whose support is concentrated to a point at the infinity; namely, supp $T = \{(1, 1, \cdots, 1)\infty\}$.

References

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