47. A Canonical Form of a System of Microdifferential Equations with Non-Involutory Characteristics and Branching of Singularities

By Toshinori ÔAKU
Department of Mathematics, University of Tokyo

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We study a system $\mathcal{M}$ of microdifferential (=pseudodifferential) equations. We assume that the characteristic variety V of $\mathcal{M}$ is the union of two regular submanifolds with non-involutory intersection. We also assume that $\mathcal{M}$ has regular singularities along V. (Precise assumptions will be given below.) In § 1, we give a canonical form of $\mathcal{M}$ in the complex domain. Applying this result, we study in § 2 the branching of supports of microfunction solutions of $\mathcal{M}$ under the additional assumption that $\mathcal{M}$ is hyperbolic. Details of this article will appear elsewhere.

§ 1. A canonical form of a system with regular singularities along its non-involutory characteristics. Let X be an n-dimensional complex manifold and T^*X be its cotangent bundle. We identify the zero section of T^*X with X. Let $z=(z_1, \cdots, z_n)$ be a local coordinate system of X. Then $(z, \langle\zeta, dz\rangle)=(z, \zeta)=(z_1, \cdots, z_n, \zeta_1, \cdots, \zeta_n)$ denotes a point of T^*X. We denote by $\mathcal{E}_X$ the sheaf on T^*X of microdifferential operators (of finite order). Note that $\mathcal{E}_X$ is denoted by $\mathcal{P}_X^f$ in [6]. Let $\mathcal{O}(j)$ be the sheaf on T^*X of holomorphic functions homogeneous of degree j with respect to the fiber coordinates. We denote by $\mathcal{E}(j)$ the sheaf of microdifferential operators of order at most j. There is a natural homomorphism

$$\sigma_j : \mathcal{E}(j)\to\mathcal{O}(j)\cong\mathcal{E}(j)/\mathcal{E}(j-1).$$

If $P \in \mathcal{E}(j)-\mathcal{E}(j-1)$, we call $\sigma(P)=\sigma_j(P)$ the principal symbol of P. For a homogeneous (=conic) involutory analytic subset V of T^*X-X, we set $I_V(j)=\{f \in \mathcal{O}(j)\,;\, f|_V=0\}$. Then $\mathcal{O}_V(0)=\mathcal{O}(0)/I_V(0)$ is a coherent sheaf of rings on V. We set $\mathcal{J}_V=\{P\in\mathcal{E}(1)\,;\,\sigma_1(P)\in I_V(1)\}$ and denote by $\mathcal{E}_V$ the subring of $\mathcal{E}_X$ generated by $\mathcal{J}_V$.

Let $\omega=\zeta_1dz_1+\cdots+\zeta_ndz_n$ be the fundamental 1-form on T^*X. A homogeneous involutory submanifold of T^*X-X is said to be regular if the pull back of ω to it vanishes nowhere.

For a submanifold W of T^*X and a point p of W, we say that p is a point of rank $2r$ in W if the rank of the skew-symmetric bilinear form $d\omega$ on T_pW is of rank $2r$. If each point of W is a point of rank $2r$ in W, we say that W is of rank $2r$ and write rank $W=2r$.

Now let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module (i.e., a system of microdifferential equations) defined on an open subset Ω of T^*X-X; let $V=V_1 \cup V_2$ be a homogeneous involutory analytic subset of Ω. We assume the following conditions:

(A.1) V_1 and V_2 are d-codimensional homogeneous regular involutory submanifolds of Ω, and $V_0=V_1\cap V_2$ is non-singular.

(A.2) V_1 and V_2 intersect normally, i.e., $T_pV_1\cap T_pV_2=T_pV_0$ for any $p\in V_0$.

(A.3) $\dim V_1=\dim V_2=\dim V_0+1$.

(A.4) $\operatorname{rank} V_1=\operatorname{rank} V_2=\operatorname{rank} V_0$.

(A.5) $\mathcal{M}$ has regular singularities along V; i.e., any coherent sub-$\mathcal{E}_V$-module of $\mathcal{M}$ that is defined on an open subset of Ω is coherent over $\mathcal{E}(0)$. (See [5], [3].)

Let p_0 be a point of V_0. We can find a neighborhood U of p_0 and a coherent sub-$\mathcal{E}_V$-module $\mathcal{M}_0$ of $\mathcal{M}|_U$ such that $\mathcal{E}_X\mathcal{M}_0=\mathcal{M}|_U$. In view of (A.5), we see that $\bar{\mathcal{M}}_0=\mathcal{M}_0/\mathcal{E}(-1)\mathcal{M}_0$ is a coherent $\mathcal{O}_V(0)$-module. We make the additional assumption:

(A.6) $\bar{\mathcal{M}}_0$ is a locally free $\mathcal{O}_V(0)$-module of rank m.

Let p be an arbitrary point in $V_0\cap U$. Then (A.6) ensures that there exist generators $u_1, \cdots, u_m$ of $\mathcal{M}_0$ over $\mathcal{E}(0)$ in a neighborhood of p whose residue classes are free generators of $\bar{\mathcal{M}}_0$ over $\mathcal{O}_V(0)$. In view of (A.1)–(A.4), we can find two microdifferential operators P_1 and P_2 in a neighborhood of p such that $\sigma(P_j)=0$ on V_j $(j=1, 2)$ and that the Poisson bracket $\{\sigma(P_1), \sigma(P_2)\}$ never vanishes on V_0. Let P_j be of order l_j and set $l=l_1+l_2$. Assumption (A.5) guarantees the existence of $A_{ij}\in\mathcal{E}(l-1)$ $(i, j=1, \cdots, m)$ defined near p such that

$$P_1P_2u_i=\sum_{j=1}^{m} A_{ij}u_j \qquad (i=1, \cdots, m).$$

Setting

$$a_{ij}(p)=\sigma_{l-1}(A_{ij})(p)/\{\sigma(P_1), \sigma(P_2)\}(p),$$

we define a polynomial e_{12} in λ by

$$e_{12}(\lambda, p, \mathcal{M}_0)=\det(\lambda I_m+(a_{ij}(p))_{1\leq i,j\leq m});$$

here I_m is the unit matrix of degree m. We can easily see that e_{12} is independent of the choice of operators P_1 and P_2, and generators $u_1, \cdots, u_m$ of $\mathcal{M}_0$ mentioned above. Thus $e_{12}(\lambda, p, \mathcal{M}_0)$ is well-defined for $p\in V_0\cap U$.

Remark. Interchanging V_1 and V_2, we can define e_{21} in the same manner and have the relation

$$e_{21}(\lambda, p, \mathcal{M}_0)=(-1)^m e_{12}(-\lambda-1, p, \mathcal{M}_0).$$

Let $\lambda=\lambda_1, \cdots, \lambda_m$ be the roots of the equation $e_{12}(\lambda, p_0, \mathcal{M}_0)=0$ in λ. For each complex number λ, we define a set $J(\lambda)$ by

$$J(\lambda)=\{j\in\{1, \cdots, m\}; \lambda_j-\lambda\in\mathbf{Z}\}.$$

Put $m(\lambda)=\sharp J(\lambda)$ (the cardinal number of $J(\lambda)$). Further, we set

$$J(0,1)=\{j\in J(0)\,;\,\lambda_j<0\},\qquad J(0,2)=\{j\in J(0)\,;\,\lambda_j\geqq 0\},$$

and $m(0,i)=\sharp J(0,i)$ for $i=1,2$. Put

$$\Lambda=\{\lambda\in \boldsymbol{C}\,;\,-1<\mathrm{Re}\,\lambda\leqq 0,\ J(\lambda)\neq\phi\}.$$

Note that

$$m=\sum_{\lambda\in\Lambda} m(\lambda),\qquad m(0)=m(0,1)+m(0,2).$$

We use the notation $D=(D_1,\cdots,D_n)$ with $D_j=\partial/\partial z_j$ and set $z'=(z_{d+1},\cdots,z_n)$ and $D'=(D_{d+1},\cdots,D_n)$.

On the above assumptions we have the following

Theorem 1. *There exists a quantized contact transformation Φ associated with a local contact transformation φ of T^*X such that $\mathcal{M}$ is isomorphic to $\Phi(\mathcal{N})$ as an $\mathcal{E}_X$-module. Here $\mathcal{N}$ is a system defined on a neighborhood of $\varphi(p_0)=(0,dz_n)$ which takes the following form:*

$$\mathcal{N}=\bigoplus_{\lambda\in\Lambda}\mathcal{N}_\lambda,$$

$$\mathcal{N}_\lambda:\ (z_1D_1I_{m(\lambda)}-A_\lambda)v_\lambda=D_2v_\lambda=\cdots=D_dv_\lambda=0\,;$$

here A_λ is an $(m(\lambda),m(\lambda))$ matrix of microdifferential operators of order at most 0 *defined near* $(0,dz_n)$, *and v_λ is a column vector of $m(\lambda)$ unknown functions. Moreover, A_λ has the form $A_\lambda=A_\lambda(z',D')$ for $\lambda\neq 0$, while A_0 has the form*

$$A_0=\begin{bmatrix}A_{11}(z',D') & A_{12}(z',D')D_1\\ z_1A_{21}(z',D') & A_{22}(z',D')\end{bmatrix}$$

with A_{ij} being an $(m(0,i),m(0,j))$ matrix of microdifferential operators. In addition, all the eigenvalues of $\sigma_0(A_\lambda)(0,dz_n)$ equal λ if $\lambda\neq 0$; all the eigenvalues of $\sigma_0(A_{11})(0,dz_n)$ are -1, and those of $\sigma_0(A_{22})(0,dz_n)$ are 0.

Remark. When $m=1$, this theorem has been proved by Kashiwara-Kawai-Oshima [4, Th. 3]. To prove Theorem 1, we use methods due to Kashiwara-Oshima [5].

§ 2. Branching of supports of microfunction solutions. Let M be an n-dimensional real analytic manifold and X be its complexification. We denote by $T^*_MX=\sqrt{-1}\,T^*M$ the conormal bundle of M in T^*X. Let $\mathcal{C}_M$ denote the sheaf on T^*_MX of microfunctions.

Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined on an open subset Ω of T^*X-X; let $V=V_1\cup V_2$ be a homogeneous involutory analytic subset of Ω. Let the assumptions (A.1)–(A.6) be satisfied. Moreover, we assume

(A.7) V_1 and V_2 are real; i.e., V_j is a complexification of the real analytic manifold $V_j^{\boldsymbol{R}}=V_j\cap T^*_MX$ for $j=1,2$.

Let p_0 be a point in $V_0^{\boldsymbol{R}}=V_1^{\boldsymbol{R}}\cap V_2^{\boldsymbol{R}}$ and let $\lambda_1,\cdots,\lambda_m$ be the roots of the equation $e_{12}(\lambda,p_0,\mathcal{M}_0)=0$ in λ. Then we also assume

(A.8) $\lambda_j\notin\{-1,-2,\cdots\}$ for $j=1,\cdots,m$.

Theorem 2. *Under the assumptions* (A.1)–(A.8), *we have*

$$\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \Gamma_{V_1^R}(\mathcal{C}_M))_{p_0}=0.$$

Moreover, if $\lambda_j \notin \boldsymbol{Z}$ *for* $j=1, \cdots, m$, *we have*

$$\mathcal{E}xt^i_{\mathcal{E}_X}(\mathcal{M}, \Gamma_{V_k^R}(\mathcal{C}_M))_{p_0}=0$$

for any $i \in \boldsymbol{Z}$ *and* $k=1, 2$.

Let us denote by $b_j(p_0)$ the bicharacteristic of V_j^R through p_0 for $j=1, 2$. Assumptions (A.1)–(A.4) and (A.7) imply that $b_1(p_0)$ and $b_2(p_0)$ intersect normally and their intersection is 1-codimensional both in $b_1(p_0)$ and in $b_2(p_0)$. Hence $b_j(p_0)-b_k(p_0)=b_j(p_0)-V_0^R$ consists of two connected components $b_j^+(p_0)$ and $b_j^-(p_0)$ in a neighborhood of p_0 for $j, k=1, 2$ and $j \neq k$. Here the choice of $b_j^+(p_0)$ and $b_j^-(p_0)$ is arbitrary.

Theorem 3. *Suppose that* (A.1)–(A.8) *are satisfied. Then there exists a fundamental neighborhood system* $\{U_l\}_{l=1,2,\dots}$ *of* p_0 *in* T_M^*X *which satisfies the following: If* f *is a microfunction solution of* $\mathcal{M}$ (i.e., a section of $\mathcal{H}om_{\mathcal{E}}(\mathcal{M}, \mathcal{C}_M)$) *defined on some* U_l *such that*

$$b_2^+(p_0) \cap U_l \not\subset \operatorname{supp} f, \qquad (b_1(p_0) \cup b_2^-(p_0)) \cap U_l \not\subset \operatorname{supp} f,$$

then f *vanishes on a neighborhood of* $(b_1(p_0) \cup b_2(p_0)) \cap U_l$.

For a subset S of T_M^*X, we set $\boldsymbol{R}^+S=\{cp\,;\, c \in \boldsymbol{R},\ c>0,\ p \in S\}$.

Theorem 4. *Let* $\mathcal{M}$ *and* V *satisfy* (A.1)–(A.7). *Then there exist a neighborhood* U' *of* p_0 *in* T_M^*X *and a microfunction solution* f *of* $\mathcal{M}$ *defined on* U' *such that*

$$\boldsymbol{R}^+b_1(p_0) \cap U' \subset \operatorname{supp} f \subset \boldsymbol{R}^+(b_1(p_0) \cup b_2^+(p_0)) \cap U'.$$

Moreover, if $\lambda_j \notin \{-1, -2, \cdots\}$ *for some* $j \in \{1, \cdots, m\}$, *we can choose* f *so that*

$$\operatorname{supp} f=\boldsymbol{R}^+(b_1(p_0) \cup b_2^+(p_0)) \cap U'.$$

Corollary. *Let* P_1 *and* P_2 *be microdifferential operators of order* l_1 *and* l_2 *respectively, defined on a neighborhood of* $p_0 \in T_M^*X - M$. *Suppose that*

$$\sigma(P_1)(p_0)=\sigma(P_2)(p_0)=0, \qquad \{\sigma(P_1), \sigma(P_2)\}(p_0) \neq 0,$$

and that $\sigma(P_1)$ *and* $\sigma(P_2)$ *are real valued on* T_M^*X. *Let* A_{ij} $(i, j=1, \cdots, m)$ *be microdifferential operators of order at most* $l-1=l_1+l_2-1$ *defined near* p_0, *and set* $A=(A_{ij})$. *We assume that no eigenvalue of the matrix* $(\sigma_{l-1}(A_{ij})(p_0)/\{\sigma(P_1), \sigma(P_2)\}(p_0))$ *is a negative integer. Then the conclusions of Theorems* 3 *and* 4 *are valid for the system*

$$(P_1P_2I_m+A)u=0$$

with $V_k=\{\sigma(P_k)=0\}$ *for* $k=1, 2$. *Here* u *is a column vector of* m *unknown functions.*

Remark. When $m=1$, an analogue in the C^∞-category of this corollary has been obtained by Ivrii [2] and Hanges [1].

Example. Set $x=(x_1, \cdots, x_n) \in \boldsymbol{R}^n$ and $D_j=\partial/\partial x_j$. We put

$$P=(D_1^2-x_1^2(D_2^2+\cdots+D_n^2))I_m+Q\,;$$

here Q is an (m, m) matrix of microdifferential operators of order at most 1 defined on a neighborhood of $p_0=(0, \sqrt{-1}\,dx_n) \in \sqrt{-1}\,T^*\boldsymbol{R}^n$.

Set $b_k(p_0)=\{(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*\boldsymbol{R}^n;\ x_n=(-1)^k x_1^2/2,\ x_2=\cdots=x_{n-1}=0,\ \xi_1=-(-1)^k x_1,\ \xi_2=\cdots=\xi_{n-1}=0,\ \xi_n=1\}$ and $b_k^{\pm}(p_0)=\{(x, \sqrt{-1}\xi) \in b_k(p_0);\ \pm x_1>0\}$ for $k=1, 2$. Assume that no eigenvalue of the matrix $\sigma_1(Q)(p_0)$ belongs to $\{\pm\sqrt{-1}, \pm 3\sqrt{-1}, \pm 5\sqrt{-1}, \cdots\}$. Let f be a column vector of m microfunctions defined on a neighborhood of p_0 such that $Pf=0$. Under these assumptions, if f vanishes on any two of half-bicharacteristics $b_1^+(p_0)$, $b_1^-(p_0)$, $b_2^+(p_0)$, and $b_2^-(p_0)$, then f vanishes on a neighborhood of p_0.

References

[1] N. Hanges: Parametrices and propagation of singularities for operators with non-involutive characteristics. Indiana Univ. Math. J., **28**, 87–97 (1979).

[2] V. Ya. Ivrii: Wave fronts of solutions of certain pseudodifferential equations. Funct. Anal. Appl., **10**, 141–142 (1976).

[3] M. Kashiwara and T. Kawai: On holonomic systems of micro-differential equations, III (to appear).

[4] M. Kashiwara, T. Kawai, and T. Oshima: Structure of cohomology groups whose coefficients are microfunction solution sheaves of systems of pseudo-differential equations with multiple characteristics. I. Proc. Japan Acad., **50**, 420–425 (1974).

[5] M. Kashiwara and T. Oshima: Systems of differential equations with regular singularities and their boundary value problems. Ann. of Math., **106**, 145–200 (1977).

[6] M. Sato, T. Kawai, and M. Kashiwara: Microfunctions and pseudo-differential equations. Lect. Notes in Math., vol. 287, Springer, Berlin-Heidelberg-New York, pp. 265–529 (1973).