118. Connected Sum along the Cycle Operation of $S^p \times \tilde{S}^{n-p}$ on π-Manifolds

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(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1981)

Let M be an n-dimensional simply-connected closed smooth π-manifold. Let p be an integer, $2 \leqq p \leqq (n/2)-2$, and let $\varphi: S^p \to M$ be an imbedding which represents a generator of $H_p(M)$. We consider the operation of the connected sum of M and $S^p \times \tilde{S}^{n-p}$ along the cycle $\varphi(S^p)$ of M and the cycle $S^p \times \{*\}$ of $S^p \times \tilde{S}^{n-p}$, where $\tilde{S}^{n-p}$ denotes a homotopy $(n-p)$-sphere. The topological structure of M is invariant under this operation. We intend to investigate the effect of this operation on the differentiable structure on M.

Let $I_p(M, \varphi)$ denote the set consisting of those homotopy $(n-p)$-spheres which operate on M along the cycle $\varphi(S^p)$ trivially. In this paper we show that $I_p(M, \varphi)$ is trivial in the case where M is a product of standard spheres and φ represents its standard spherical cycle. As an application we show that $S^2 \times S^3 \times \tilde{S}^{10}$ is not diffeomorphic to $S^2 \times S^3 \times S^{10}$, where $\tilde{S}^{10}$ represents a generator of the group of homotopy 10-spheres $\Theta_{10} \approx Z_6$. This result is in contrast with the fact that $S^5 \times \tilde{S}^{10}$ is diffeomorphic to $S^5 \times S^{10}$. Throughout this paper, we mean by a diffeomorphism an orientation preserving diffeomorphism unless otherwise stated.

Details and further arguments will appear elsewhere.

1. Let M be an n-dimensional simply-connected closed smooth π-manifold and $\varphi: S^p \to M$ an imbedding which represents a generator of $H_p(M)$, $p \geqq 2$. Let $\tilde{S}^{n-p} = D_1^{n-p} \bigcup_{\gamma} D_2^{n-p}$ for $\gamma \in \mathrm{Diff}\,(S^{n-p-1})$ and define an imbedding $\iota: S^p \to S^p \times \tilde{S}^{n-p}$ by $\iota(x) = (x, 0)$, where $0 \in D_2^{n-p}$ denotes the center of the $(n-p)$-disk D_2^{n-p}. A trivialization of a tubular neighborhood of $\iota(S^p)$ can be defined by the composition of the maps: $S^p \times D^{n-p} \xrightarrow{1\times\eta} S^p \times D_2^{n-p} \hookrightarrow S^p \times \tilde{S}^{n-p}$, where $\eta: D^{n-p} \to D_2^{n-p}$ is an orientation reversing diffeomorphism. We denote this trivialization by $I: S^p \times D^{n-p} \to S^p \times \tilde{S}^{n-p}$.

It is known that $S^p \times \tilde{S}^{n-p}$ is diffeomorphic to $S^p \times S^{n-p}$ if $(n-p)-p \leqq 3$. (See Hsiang, Levine and Szczarba [2].) Therefore it suffices to consider the case that $(n-p)-p \geqq 4$, that is, $2 \leqq p \leqq (n/2)-2$.

Since M is a π-manifold, the tubular neighborhood of the imbedded p-dimensional sphere $\varphi(S^p)$ is trivial. We denote this trivialization by $\Phi: S^p \times D^{n-p} \to M$ (we choose Φ such that it preserves orientation). Thus we obtain from the disjoint sum

$$(M-\varphi(S^p))+(S^p \times \tilde{S}^{n-p}-\iota(S^p))$$

a *connected sum of* M *and* $S^p \times \tilde{S}^{n-p}$ *along the cycle* $\varphi(S^p)$ of M and the standard p-cycle $\iota(S^p)$ of $S^p \times \tilde{S}^{n-p}$ by identifying $\Phi(u, t\cdot v)$ with $I(u, (1-t)\cdot v)$ for each $u \in S^p$, $v \in \partial D^{n-p}=S^{n-p-1}$ and t, $0<t<1$ (cf. Novikov [6]). We denote this connected sum by

$$(M, \Phi, S^p) \mathbin{\#_p} (S^p \times \tilde{S}^{n-p}, I, S^p)$$

which is naturally homeomorphic to M. We choose the orientation for $(M, \Phi, S^p) \mathbin{\#_p} (S^p \times \tilde{S}^{n-p}, I, S^p)$ which is compatible with that of M and $S^p \times \tilde{S}^{n-p}$.

For another trivialization I' of $\iota(S^p)$, we can choose the suitable trivialization Φ' of $\varphi(S^p)$ such that $(M, \Phi', S^p) \mathbin{\#_p} (S^p \times \tilde{S}^{n-p}, I', S^p)$ is equal to $(M, \Phi, S^p) \mathbin{\#_p} (S^p \times \tilde{S}^{n-p}, I, S^p)$. Therefore we fix the trivialization I of $\iota(S^p)$ and denote $(M, \Phi, S^p) \mathbin{\#_p} (S^p \times \tilde{S}^{n-p}, I, S^p)$ by $(M, \Phi, S^p) \mathbin{\#_p} S^p \times \tilde{S}^{n-p}$.

Obviously we have the following

Lemma 1. *If the imbeddings* $\varphi_0, \varphi_1: S^p \to M$ *are homotopic, then there exists a diffeomorphism* $H: M \to M$ *such that* $H \circ \varphi_1 = \varphi_0$ *and* H *is isotopic to the identity map of* M.

Therefore the imbedding φ can be replaced by any homotopic imbedding.

Lemma 2. *If* Φ *and* Φ' *are trivializations of a tubular neighborhood of* $\varphi(S^p)$ *in* M, *then* $(M, \Phi, S^p) \mathbin{\#_p} S^p \times \tilde{S}^{n-p}$ *and* $(M, \Phi', S^p) \mathbin{\#_p} S^p \times \tilde{S}^{n-p}$ *are diffeomorphic modulo a point. Moreover, the diffeomorphism modulo a point can be chosen so that it is an identity map on* $M-\Phi(S^p \times D^{n-p})$.

Note that this difference at one point represents the Milnor-Munkres-Novikov pairing $\tau_{n-p,p}(\tilde{S}^{n-p} \otimes \alpha)$ where α denotes the homotopy class of $\Phi^{-1} \circ \Phi'$. (See Kawakubo [4] and De Sapio [8].)

Lemma 3. $(M, \Phi, S^p) \mathbin{\#_p} S^p \times \tilde{S}^{n-p}$ *is a* π-*manifold.*

Let $H_{n-p,p}$ denote the subgroup of Θ_{n-p} consisting of those homotopy $(n-p)$-spheres $\tilde{S}^{n-p}$ such that $S^p \times \tilde{S}^{n-p}$ is diffeomorphic to $S^p \times S^{n-p}$, and denote the quotient group $\Theta_{n-p}/H_{n-p,p}$ by $K_{n-p,p}$. It is known that $H_{n-p,p}$ is related to the π-imbedding of homotopy spheres in Euclidean space and some of them are determined. (See Katase [3], Bandō and Katase [1], Hsiang, Levine and Szczarba [2] and Levine [5].) Note that, in general, the elements of $K_{n-p,p}$ are not in one to one correspondence with the diffeomorphism classes of $S^p \times \tilde{S}^{n-p}$. (See De Sapio [7].)

From Lemma 5 of De Sapio [7] we obtain the following

Lemma 4. *If $\tilde{S}_1^{n-p}$ and $\tilde{S}_2^{n-p}$ represent the same element of $K_{n-p,p}$, then there exists a diffeomorphism $f: S^p\times\tilde{S}_1^{n-p}\to S^p\times\tilde{S}_2^{n-p}$ such that f maps $S^p\times 0_1$ in $S^p\times\tilde{S}_1^{n-p}$ identically onto $S^p\times 0_2$ in $S^p\times\tilde{S}_2^{n-p}$ and $f(S^p\times D_{1,2}^{n-p})=S^p\times D_{2,2}^{n-p}$, where $\tilde{S}_i^{n-p}=D_{i,1}\bigcup_{\gamma_i} D_{i,2}^{n-p}$ and 0_i is the center of $D_{i,2}^{n-p}$ $(i=1,2)$.*

Now we regard the operation of an element $[\tilde{S}^{n-p}]$ of $K_{n-p,p}$ on M along the cycle $\varphi(S^p)$ as a connected sum of M and $S^p\times\tilde{S}^{n-p}$ along the cycle $\varphi(S^p)$ and the cycle $\iota(S^p)$. We call this operation a *$K_{n-p,p}$-operation on M along the cycle $\varphi(S^p)$.*

Theorem 1. *The $K_{n-p,p}$-operation on M along the cycle $\varphi(S^p)$ is well defined, commutative, associative up to a diffeomorphism modulo a point.*

Remark. If $\tilde{S}^{10}$ is a generator of $\Theta_{10}\approx \boldsymbol{Z}_6$, it is known that $S^3\times\tilde{S}^{10}\ \#\ \tilde{S}^{13}$ is diffeomorphic to $S^3\times\tilde{S}^{10}$ for all $\tilde{S}^{13}\in\Theta_{13}\approx\boldsymbol{Z}_3$, that is, the inertia group of $S^3\times\tilde{S}^{10}$ is equal to Θ_{13}. (See Kawakubo [4] and De Sapio [8].) Therefore, even if $\tilde{S}^{13}$ is a non-trivial element of Θ_{13}, $S^2\times(S^3\times\tilde{S}^{10}\ \#\ \tilde{S}^{13})=S^2\times S^3\times\tilde{S}^{10}\ \#_2\ S^2\times\tilde{S}^{13}$ is diffeomorphic to $S^2\times S^3\times\tilde{S}^{10}$. However, in case $\tilde{S}^{13}$ is a non-trivial element, $S^2\times\tilde{S}^{13}$ is not diffeomorphic to $S^2\times S^{13}$. Therefore this gives an example of a manifold on which non-trivial element of $K_{13,2}$ operates trivially.

2. Let $I_p(M,\varphi)$ denote the subset of $K_{n-p,p}$ consisting of those classes of homotopy $(n-p)$-spheres $[\tilde{S}^{n-p}]$ such that $(M,\Phi,S^p)\ \#_p\ S^p\times\tilde{S}^{n-p}$ is diffeomorphic to M modulo a point. $I_p(M,\varphi)$ plays an important role in differentiable structures on M.

Remarks. 1) It is not known whether or not there exist a manifold and an imbedding φ such that $I_p(M,\varphi)$ does not form a group.

2) Obviously the following is a sufficient condition for $I_p(M,\varphi)$ to form a group: For any imbedding $\varphi': S^p\to M$ which represents a generator of $H_p(M)$, there exists a diffeomorphism $f: M\to M$ such that $f\circ\varphi'=\varphi$.

3) In the case in which M is a $(p-1)$-connected manifold and φ represents a generator of $H_p(M)\approx\boldsymbol{Z}$, $I_p(M,\varphi)$ forms a group. In this case we have a few results on $I_p(M,\varphi)$.

In the case where M is a product of standard spheres, we have the following

Theorem 2. *If $M=S^p\times S^{q_1}\times\cdots\times S^{q_j}$ $(p, q_k\geqq 2$ and $p+\sum_{k=1}^{j} q_k=n)$ and if $\varphi: S^p\to M$ is an imbedding given by $x\mapsto(x,\{*\})$, then $I_p(M,\varphi)$ is trivial.*

Proof. If $\tilde{S}^{n-p}\in I_p(M,\varphi)$, then there exists a diffeomorphism

$$h: M\ \#_p\ S^p\times\tilde{S}^{n-p}\ \#\ \tilde{S}^n\longrightarrow M$$

for some $\tilde{S}^n \in \Theta_n$ and it follows from Corollary 1.5 of Schultz [9] that we can assume the induced isomorphism h_* of homology groups to be equal to the isomorphism by the natural homeomorphism. Imbed $S^{q_1}\times\cdots\times S^{q_j}$ in the interior of D^{n-p+1}. Then M is naturally imbedded in $S^p\times D^{n-p+1}$, and M divides $S^p\times D^{n-p+1}$ into two components: the inside W_1 and the outside W_2. Consider a certain connected sum W_2' of $S^p\times\tilde{S}^{n-p}\times I$ and $\tilde{S}^n\times I$ to W_2 such that $\partial W_2'=(-(M \# S^p\times\tilde{S}^{n-p} \# \tilde{S}^n)) \cup(S^p\times\tilde{S}^{n-p} \# \tilde{S}^n)$, and define $W=W_1\cup_h W_2'$. Then $\partial W \overset{p}{=} S^p\times\tilde{S}^{n-p} \# \tilde{S}^n$. On the other hand, by a standard argument involving van Kampen's theorem, Mayer-Vietoris exact sequences, a theorem of J. H. C. Whitehead and Theorem 4.1 of Smale [10], it follows that W is diffeomorphic to $S^p\times D^{n-p+1}$, and hence $\partial W=S^p\times\tilde{S}^{n-p} \# \tilde{S}^n$ is diffeomorphic to $S^p\times S^{n-p}$. Therefore it follows from Corollary 3 of Katase [3] that $\tilde{S}^{n-p}$ represents a trivial element of $K_{n-p,p}$. Q.E.D.

Remark. It follows from this theorem and the previous example that $S^2\times S^3\times\tilde{S}^{10}$ is not diffeomorphic to $S^2\times S^3\times S^{10}$.

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