115. *Isomonodromy Problem of Schlesinger Equations*

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§1. **Introduction.** In this article we study the monodromy preserving deformation of linear ordinary differential equations with regular singular points. L. Schlesinger [1] is one of pioneers in this field and established a general framework of the theory, which was recently extended by M. Jimbo, T. Miwa, K. Ueno [4], [5] and B. Klares [3] to the case admitting irregular singular points.

The case treated by the formers is the following

$$(1.1)\qquad \frac{dY}{dx}=A(x)Y,\qquad A(x)=\sum_{\nu=1}^{n}\sum_{k=0}^{r_\nu}\frac{A_{\nu,-k}}{(x-a_\nu)^{k+1}}-\sum_{k=1}^{r_\infty}A_{\infty,-k}x^{k-1}$$

$$(1.2)\qquad A_{\mu,-r_\mu}=G^{(\mu)}T^{(\mu)}_{-r_\mu}G^{(\mu)-1},\quad A_{\infty 0}=-\sum_{\nu=1}^{n}A_{\nu 0}\quad (\mu=1,\cdots,n,\infty).$$

(A.I) $T^{(\mu)}_{-r_\mu}$: diagonal with eigenvalues mutually distinct (if $r_\mu\geq 1$)

(A.II) : distinct modulo integers (if $r_\mu=0$).

L. Schlesinger accomplished the deformation theory under the assumption (A.II), which we are to relieve in the following. The assumptions (A.I), (A.II) need to be removed indeed since many examples in both mathematics and physics are not equipped with them. The assumption (A.I) thus will be also taken away in a forthcoming note by applying the results in this article.

In §§2 and 3, we investigate the structure of solutions in our case and execute the deformation theory without the assumption (A.II), respectively. Detailed discussion will be published elsewhere.

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§2. **The structure of solutions.** We consider the following system of Fuchsian class containing parameters a_ν that are the positions of singularities

$$(2.1)\qquad \frac{dY}{dx}=A(x,a)Y,\quad A(x,a)=\sum_{\nu=1}^{n}\frac{A_\nu(a)}{x-a_\nu},\quad a=(a_1,\cdots,a_n)$$
$$(a_i\neq a_j\qquad \text{if } i\neq j)$$

where $A_\nu(a)$ are $m\times m$ matrices and assumed to be holomorphic in a. We set $A_\infty=-\sum_{\nu=1}^{n}A_\nu$. Let the eigenvalues of A_μ be as follows:

$$(2.2)\qquad \{\lambda_1^\mu,\cdots,\lambda_{q_\mu}^\mu\},\quad \lambda_i^\mu\neq\lambda_j^\mu\quad \text{if } i\neq j.$$

We denote by $m_i^\mu=[R_e\lambda_i^\mu]$ the greatest integers not exceeding $R_e\lambda_i^\mu$ and set $\tilde{\lambda}_i^\mu=\lambda_i^\mu-m_i^\mu$, hence $0\leq R_e\tilde{\lambda}_i^\mu<1$. Suppose that m_i^μ are constant. Further are assumed that

(2.3) the degrees of the elementary divisors of A_μ are fixed.

We remark that under the assumptions (2.2), (2.3) the eigenvalues $\lambda_i^\mu(a)$ are holomorphic in a and that there exist holomorphic matrices $G^{(\mu)}(a)$, $G^{(\mu)}(a)^{-1}$ such that $G^{(\mu)}\tilde{A}_\mu G^{(\mu)-1}=A_\mu$, where $\tilde{A}_\mu$ are the Jordan normal forms of A_μ (W. Wasow [9]). Without loss of generality, we may assume that $A_\infty=\tilde{A}_\infty$, $G^{(\infty)}=1$ and that the matrices $\tilde{A}_\mu$ have the forms:

(2.4) $$\tilde{A}_\mu=\text{block diag}\{\lambda_1^\mu E_1^\mu+H_1^\mu,\ \cdots,\ \lambda_{r_\mu}^\mu E_{r_\mu}^\mu+H_{r_\mu}^\mu\}$$

where

(2.5) $$m_1^\mu\geq m_2^\mu\geq\cdots\geq m_{r_\mu}^\mu \qquad (\mu=1,\ \cdots,\ n,\ \infty).$$

Here E_i^μ and H_i^μ denote the identity and shifting matrices, respectively. Observe that $r_\mu\geq q_\mu$. When $\lambda_i^\mu-\lambda_j^\mu$ $(1\leq i<j\leq r_\mu)$ is a positive integer we define l_{ij}^μ as follows:

(2.6) $$l_{ij}^\mu=\lambda_i^\mu-\lambda_j^\mu=m_i^\mu-m_j^\mu \qquad (\geq 1).$$

Instead of (2.1), we consider the following equivalent systems

(2.7) $$\frac{dY^{(\mu)}}{dx}=A^{(\mu)}(x,a)Y^{(\mu)}, \qquad A^{(\mu)}(x,a)=G^{(\mu)}(a)^{-1}A(x,a)G^{(\mu)}(a)$$
$$(\mu=1,\ \cdots,\ n,\ \infty).$$

Proposition (F. Gantmacher [8]). *Every system* (2.7) *has a solution* $Y^{(\mu)}$ *at* $x=a_\mu$ *of the following representation*

(2.8) $$Y^{(\mu)}(x,a)=\tilde{Y}^{(\mu)}(x,a)z_\mu^{M^{(\mu)}}z_\mu^{T_0^{(\mu)}(a)} \qquad (\mu=1,\ \cdots,\ n,\ \infty)$$

where

$$\tilde{Y}^{(\mu)}(x,a)=1+\sum_{k=1}^{\infty}Y_k^{(\mu)}(a)z_\mu^k$$

$$M^{(\mu)}=\text{block diag}\{m_1^\mu E_1^\mu,\ \cdots,\ m_{r_\mu}^\mu E_{r_\mu}^\mu\}$$

(2.9) $$T_0^{(\mu)}(a)=\begin{pmatrix}\tilde{\lambda}_1^\mu(a)E_1^\mu+H_1^\mu, & B_{12}^\mu(a), & \cdots, & B_{1r_\mu}^\mu(a)\\ 0, & \tilde{\lambda}_2^\mu(a)E_2^\mu+H_2^\mu, & \cdots, & B_{2r_\mu}^\mu(a)\\ \cdots & \cdots & \cdots & \cdots\\ 0, & 0, & \cdots, & \tilde{\lambda}_{r_\mu}^\mu(a)E_{r_\mu}^\mu+H_{r_\mu}^\mu\end{pmatrix}$$
$$=\tilde{T}_0^{(\mu)}+B^{(\mu)}, \qquad (\tilde{T}_0^{(\mu)}=\text{block diag}\{\tilde{\lambda}_1^\mu E_1^\mu,\ \cdots,\ \tilde{\lambda}_{r_\mu}^\mu E_{r_\mu}^\mu\})$$

$B_{ij}^\mu(a)=0$ *if* $\lambda_i^\mu-\lambda_j^\mu$ *is not a positive integer.*

Here $z_\mu=x-a_\mu$ *if* $\mu\neq\infty$ *or* $z_\mu=x^{-1}$ *if* $\mu=\infty$.

We remark that $Y_k^{(\mu)}(a)$ and $T_0^{(\mu)}(a)$ are holomorphic in a.

The solution $Y^{(\infty)}=Y$ of the equation (2.1) can be analytically continued to a neighborhood of each singular point $x=a_\mu$ and has there the monodromy matrix

(2.10) $$M_\mu(a)=C^{(\mu)}(a)^{-1}e^{2\pi iT_0^{(\mu)}(a)}C^{(\mu)}(a)$$

where the connection matrix $C^{(\mu)}$ is defined by

(2.11) $$Y=G^{(\mu)}Y^{(\mu)}C^{(\mu)} \qquad (\mu=1,\ \cdots,\ n,\ \infty\ ;\ C^{(\infty)}=1).$$

§ 3. **Monodromy preserving deformation.** Let d be the exterior differentiation with respect to a.

Under the situation that the monodromy data:

$$T_0^{(\mu)}, \quad C^{(\mu)} \quad (\mu=1, \cdots, n, \infty) \tag{3.1}$$

being independent of a, in particular, the monodromy matrices M_μ being so by (2.10), what are the a-equations satisfied by the coefficient matrices A_ν? This is just the reason why the problem is called "monodromy preserving deformation". The answer is the Schlesinger equations also in this case.

Theorem 1. *The monodromy data remain constant,* i.e.,

$$dT_0^{(\mu)}=0, \quad dC^{(\nu)}=0 \quad (\mu=1, \cdots, n, \infty\,;\, \nu=1, \cdots, n) \tag{3.2}$$

if and only if $Y^{(\mu)}$ and $G^{(\nu)}$ satisfy the total differential equations

$$dY^{(\mu)}=\Omega^{(\mu)}Y^{(\mu)} \qquad (\mu=1, \cdots, n, \infty) \tag{3.3}$$

$$dG^{(\nu)}=\theta^{(\nu)}G^{(\nu)} \qquad (\nu=1, \cdots, n). \tag{3.4}$$

Here

$$\Omega^{(\infty)}(x, a) = -\sum_{\nu=1}^{n} \frac{A_\nu(a)da_\nu}{x-a_\nu} \tag{3.5}$$

$$\Omega^{(\nu)}(x, a)=G^{(\nu)}(a)^{-1}(\Omega^{(\infty)}(x, a)-\theta^{(\nu)}(a))G^{(\nu)}(a) \tag{3.6}$$

$$\theta^{(\nu)}(a)= \sum_{\lambda(\neq\nu)} A_\lambda(a)\frac{d(a_\nu-a_\lambda)}{a_\nu-a_\lambda}. \tag{3.7}$$

Theorem 2. *The monodromy data are independent of a if and only if $A^{(\mu)}$ and $G^{(\nu)}$ satisfy the non-linear differential equations with* (3.10)

$$dA^{(\mu)}=\frac{\partial\Omega^{(\mu)}}{\partial x}+[\Omega^{(\mu)}, A^{(\mu)}] \qquad (\mu=1, \cdots, n, \infty) \tag{3.8}$$

$$dG^{(\nu)}=\theta^{(\nu)}G^{(\nu)} \qquad (\nu=1, \cdots, n) \tag{3.9}$$

$$(\Psi^{(\mu)}_{l^\mu_{ij}})_{ij}=0 \qquad (1\leq i<j\leq r_\mu). \tag{3.10}$$

Here $\Omega^{(\mu)}$, $\theta^{(\nu)}$ are the same ones in Theorem 1 and $(\Psi^{(\mu)}_{l^\mu_{ij}})_{ij}$ imply the (i, j)-th blocks of $\Psi^{(\mu)}_{l^\mu_{ij}}$, partitioned into blocks according to the representations (2.9) *of $T_0^{(\mu)}$, that are defined through* (3.11) *with* (3.12)

$$\begin{aligned}\Psi^{(\mu)}(x, a)=\sum_{l=1}^{\infty} \Psi_l^{(\mu)}(a)z_\mu^l&= \tilde{Y}^{(\mu)}(x, a)^{-1}d\tilde{Y}^{(\mu)}(x, a)\\ &\quad+d'T^{(\mu)}(x, a)-\tilde{Y}^{(\mu)}(x, a)^{-1}\Omega^{(\mu)}(x, a)\tilde{Y}^{(\mu)}(x, a)\end{aligned} \tag{3.11}$$

$$d'T^{(\mu)}(x, a)=\begin{cases} -\{M^{(\mu)}+z_\mu^{M^{(\mu)}}T_0^{(\mu)}(a)z_\mu^{-M^{(\mu)}}\}z_\mu^{-1}da_\mu & (\mu\neq\infty)\\ 0 & (\mu=\infty).\end{cases} \tag{3.12}$$

We note that (3.8) are equivalent to the following completely integrable systems called Schlesinger equations

$$dA_\nu=-\sum_{\lambda(\neq\nu)} [A_\nu, A_\lambda]\frac{d(a_\nu-a_\lambda)}{a_\nu-a_\lambda} \qquad (\nu=1, \cdots, n). \tag{3.13}$$

Remark 1. The conditions (3.10) are characteristic of our case, indeed under the assumption (A.II) $\Psi^{(\mu)}$ themselves can be shown to vanish identically from (3.8), (3.9) ([5]).

Remark 2. A slight modification enables us to apply the notions introduced in [5], [6] to our case and draw the same results. The notions are τ-function, Schlesinger transformation and spectrum preserving deformation.

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