23. On the Inducing of Unipotent Classes for Semisimple Algebraic Groups. II

Case of Classical Type

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Let G be a connected semisimple algebraic group over an algebraically closed field K, and let $C'=\mathrm{Ind}_{L,P}^{G} C$ be as in the previous paper [10]. In this paper, we give a simple method to determine the induced class C' from C when G is of classical type. The idea is the same as that given in [9, §§4–5] to treat the Richardson classes, and based only on two well known fundamental results cited in §1. In this paper, we only assume that the characteristic p of K is zero or a good prime for G. Because of this assumption, we may work on the Lie algebra version of the inducing (for type B, C or D, using the Cayley transform in [11, 3.14] for example). Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{p}$ its parabolic subalgebra, $\mathfrak{l}$ a Levi subalgebra and $\mathfrak{n}$ the nilpotent radical of $\mathfrak{p}$. For a nilpotent class C of $\mathfrak{l}$, the induced class $C'=\mathrm{Ind}_{\mathfrak{l},\mathfrak{p}}^{\mathfrak{g}} C$ is defined as the unique class which intersects $C+\mathfrak{n}$ densely.

§1. We list up here two fundamental facts on unipotent or nilpotent classes, which play decisive roles in our method. Assume that G be simple from now on. Let $X\in\mathfrak{g}$ be nilpotent and put $G(X)=\{\mathrm{Ad}(g)X; g\in G\}$. Then it is convenient for us to use as a parameter of the class $G(X)$ the Jordan normal form of X. For types A_n, B_n, C_n and D_n, we put $N=n+1$, $2n+1$, $2n$ and $2n$ respectively. Let X be conjugate under $G_A=SL(N,K)$ to $J(p_1)\oplus J(p_2)\oplus\cdots\oplus J(p_s)$, $p_1\geqslant p_2\geqslant\cdots\geqslant p_s\geqslant 0$, $p_1+p_2+\cdots+p_s=N$, where $J(p)$ is the $p\times p$ Jordan matrix with entries 1 just above the diagonal and zero except there. We say that X and its class $G(X)$ are both *of Jordan type* $\alpha=(p_1, p_2, \cdots, p_s)$, and the latter is also denoted as $O_G(\alpha)$, when α determines the class uniquely. For type D_n with n even, if all p_i's are even, exactly two classes correspond to the same α. In this case we denote by $O_G(\alpha)$ the union of these two classes.

We realize $\mathfrak{g}$ of type B_n, D_n or C_n as a subalgebra of $\mathfrak{g}_A=\mathfrak{sl}(N,K)$ consisting of $X\in\mathfrak{g}_A$ such that $XJ+J^tX=0$ for $J=L_N$ or

$$J=\begin{pmatrix} 0_n & L_n \\ -L_n & 0_n \end{pmatrix} \quad \text{with} \quad L_n=\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \text{ (type } n\times n).$$

Here 0_n denotes the zero matrix of degree n. Then,

Theorem A [11, pp. 257–258]. *Let* $\mathfrak{g}$ *be of type* B_n *or* D_n (resp. of type C_n). *For* $X \in \mathfrak{g}_A$, *let* $G_A(X)$ *be its conjugacy class under* G_A. *Then* $G_A(X) \cap \mathfrak{g}$ *is empty or equal to* $O_G(\alpha)$ *for the Jordan type* α *of* X. *It is not empty if and only if* $\alpha=(p_1, p_2, \cdots, p_s)$ *satisfies the condition* (BD1) (resp. (C1)) *given below. Let* $r_j = \sharp\{i\,;\, p_i = j\}$ *for* $j \geqslant 1$, *then*

(BD1) r_j *is even for even* j; *and* (C1) r_j *is even for odd* j.

We define the dual partition $\alpha^\vee = [n_1, n_2, \cdots, n_t]$ of α as the partition of N given by

$$n_i = r_i + r_{i+1} + \cdots + r_t \qquad (1 \leqslant i \leqslant t),$$

if $r_j = 0$ for $j > t$. We call a Jordan type α a *G-Jordan type* if it satisfies (BD1) or (C1) according to the type of G. Denote by Jor_G the set of all G-Jordan types (for a fixed N).

For a subset E of $\mathfrak{g}$, we denote by $\mathrm{Cl}(E)$ the (Zariski) closure of E, and by $\mathrm{Jor}_G(E)$ the set of G-Jordan types α (for N) such that $O_G(\alpha) \subset E$, and put $G(E) = \{\mathrm{Ad}(g)X\,;\, X \in E,\, g \in G\}$. For two G-Jordan types α, α' for N, we define $\alpha \succcurlyeq_G \alpha'$, if $\mathrm{Cl}(O_G(\alpha)) \supset O_G(\alpha')$. If $G = G_A$, we denote $\succcurlyeq_G$ also by $\succcurlyeq_A$. Note that $\alpha \succcurlyeq_G \alpha'$ implies $\alpha \succcurlyeq_A \alpha'$. (The converse is also true as is proved using Theorem B below. But this is not necessary in the following.) Any nilpotent class in $\mathfrak{g}$ (under G) is open in its closure. So the relation $\succcurlyeq_G$ is actually a partial order, and for a G-Jordan type α,

$$\mathrm{Jor}_G(\mathrm{Cl}(O_G(\alpha))) = \{\alpha'\,;\, \alpha \succcurlyeq_G \alpha'\} = \mathrm{Jor}_G(\alpha) \quad \text{(put)}.$$

Note that $\mathrm{Jor}_G(\alpha)$ contains a unique maximal element α. For $G = G_A$, we denote $\mathrm{Jor}_G(\alpha)$ also by $\mathrm{Jor}_A(\alpha)$. Then we have $\mathrm{Jor}_G \cap \mathrm{Jor}_A(\alpha) \supset \mathrm{Jor}_G(\alpha)$ for a G-Jordan type α. (By the remark above, this inclusion is actually an equality.)

Theorem B [8]. *Let* $\alpha=(p_1, p_2, \cdots, p_s)$, $\alpha'=(p'_1, p'_2, \cdots, p'_{s'})$ *be two Jordan types for* N. *Then* $\alpha \succcurlyeq_A \alpha'$ *if and only if*

$$p_1 + p_2 + \cdots + p_j \geqslant p'_1 + p'_2 + \cdots + p'_j \qquad \textit{for } j \geqslant 1,$$

where we put $p_j = 0$ *for* $j > s$, *and similarly for* p'_j.

§2. Type A. Let $\alpha = (p_1, p_2, \cdots, p_s)$ be a Jordan type for N. We call the dual partition $\alpha^\vee = [n_1, n_2, \cdots, n_t]$ the *parabolic type* associated to α. Let $\delta = [d_1, d_2, \cdots, d_u]$ be an ordered partition of N. Denote by $\mathfrak{l}[\delta]$ and $\mathfrak{n}[\delta]$ the subalgebras of $\mathfrak{g}_A = \mathfrak{sl}(N, K)$ consisting of elements X of the following forms respectively:

$$\begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_u \end{pmatrix}, \qquad \begin{pmatrix} 0_{d_1} & & & * \\ & 0_{d_2} & & \\ & & \ddots & \\ 0 & & & 0_{d_u} \end{pmatrix}$$

where $x_i \in \mathfrak{gl}(d_i, K)$. Then $\mathfrak{p}[\delta] = \mathfrak{l}[\delta] + \mathfrak{n}[\delta]$ is a parabolic subalgebra called of type δ. We have $\mathrm{Cl}(O_{G_A}(\alpha)) = G_A(\mathfrak{n}[\alpha^\vee])$.

The subalgebra $\mathfrak{l}[\delta]$ is isomorphic to $\{(x_i) \in \prod_{1\leqslant i\leqslant u} \mathfrak{gl}(d_i, K);\ \sum_{1\leqslant i\leqslant u} \mathrm{tr}\,(x_i)=0\}$, under $X \mapsto (x_i)$. By this isomorphism, a conjugacy class C in $\mathfrak{l}[\delta]$ of nilpotent elements X is parametrized by $(\beta_1, \beta_2, \cdots, \beta_u)$ and denoted as $C(\beta_1, \beta_2, \cdots, \beta_u)$, where β_i denotes the Jordan type (for d_i) of x_i. Let $\beta_i^\vee$ be the parabolic type associated to β_i, and denote by $\mathrm{Ind}\,[\beta_1^\vee, \beta_2^\vee, \cdots, \beta_u^\vee]$ the parabolic type (for $N=d_1+d_2+\cdots+d_u$) obtained by simply arranging $\beta_1^\vee, \beta_2^\vee, \cdots$ in this order. We have the following.

Theorem 1. *Let $\mathfrak{g}=\mathfrak{g}_A=\mathfrak{sl}(N, K)$, and δ be a partition of N. Let $\mathfrak{p}=\mathfrak{p}[\delta]$ be the parabolic subalgebra of type δ, and put $\mathfrak{l}=\mathfrak{l}[\delta]$. Let C be a conjugacy class $C(\beta_1, \beta_2, \cdots, \beta_u)$ in $\mathfrak{l}$ with Jordan type $(\beta_1, \beta_2, \cdots, \beta_u)$. Then the parabolic type of the induced class $C'=\mathrm{Ind}_{\mathfrak{l},\mathfrak{p}}^{\mathfrak{g}}\, C$ is given by $\mathrm{Ind}\,[\beta_1^\vee, \beta_2^\vee, \cdots, \beta_u^\vee]$, and its Jordan type is given by $[\mathrm{Ind}\,[\beta_1^\vee, \beta_2^\vee, \cdots, \beta_u^\vee]]^\vee$.*

We remark here that this theorem is also proved by T. Tanisaki.

§ 3. Type B, C or D. Now let G be of type B_n, C_n or D_n. Let $\gamma=(q_1, q_2, \cdots, q_v)$ be any $(G_A$-$)$Jordan type for $N: q_1\geqslant q_2\geqslant \cdots \geqslant q_v$, $q_1+q_2+\cdots+q_v=N$. We define an operation T_G on γ to get a G-Jordan type from it. For G of type B or D, we define $T_G=T_{BD}$ as follows. Let $T_{BD}\gamma=\alpha=(p_1, p_2, \cdots, p_s)$.

(BDi) If q_j is *odd*, put $p_j=q_j$.

(BDii) Let q_j be *even*, and suppose p_l's have been already defined for all $l<j$. Put $I=\{i\,;\, i>j,\ q_i=q_j\}$. (Case 1) If $\sharp I$ is even, then we put $p_i=q_i$ for all $i \in I$. (Case 2) Suppose $\sharp I$ is odd. Let k be the biggest element in I, and q_m the first *even* number after q_k (q_i are all *odd* for $k<i<m$). Then we put $p_k=q_k-1$, $p_m=q_m+1$, $p_i=q_i$ for $i \in I$, $\neq k$. Adding $q_{v+1}=0$ if necessary, we repeat this process until the end.

If G is of type C, we define $\alpha=T_G\gamma=T_C\gamma$ by the processes (Ci) and (Cii), which are obtained from (BDi) and (BDii) by replacing *even* and *odd* in italic by *odd* and *even* respectively. Note that T_G is an extension of the Spaltenstein mapping in [4, p. 225] in the case of Richardson classes. The following is proved by using Theorem B.

Lemma. *Let γ be any $(G_A$-$)$Jordan type for N. If $\gamma \succcurlyeq_A \alpha$ for a G-Jordan type α, then $T_G\gamma \succcurlyeq_A \alpha$.*

For any parabolic type δ, we denote by $(T_G \circ \vee)\delta$ or by $\delta^{\vee G}$ the Jordan type $T_G(\delta^\vee)$.

We call an ordered partition $\delta=[d_1, d_2, \cdots, d_{2u-1}]$ of N a *G-parabolic type* if it satisfies $d_i=d_{2u-i}$ $(1\leqslant i\leqslant u-1)$ $(d_u=0$ is admitted). We denote it as $\delta_G=[d_1, d_2, \cdots, d_{u-1}; d_u]$. We put $\mathfrak{l}[\delta_G]=\mathfrak{g}\cap\mathfrak{l}[\delta]$, $\mathfrak{n}[\delta_G]=\mathfrak{g}\cap\mathfrak{n}[\delta]$ for the subalgebras $\mathfrak{l}[\delta]$, $\mathfrak{n}[\delta]$ of $\mathfrak{g}_A$. Then $\mathfrak{p}[\delta_G]=\mathfrak{l}[\delta_G]+\mathfrak{n}[\delta_G]$ is a parabolic subalgebra of $\mathfrak{g}$, and an element $X \in \mathfrak{l}[\delta_G]$ is a blockwise diagonal matrix $\mathrm{diag}\,(x_1, x_2, \cdots, x_{u-1}, x_u, y_{u-1}, \cdots, y_1)$, where $x_i \in \mathfrak{gl}(d_i, K)$ for $1\leqslant i\leqslant u-1$,

$x_u \in \mathfrak{so}(d_u, K)$ or $\in \mathfrak{sp}(d_u/2, K)$ according to the type of G, and $y_i = -L_{d_i}{}^t x_i L_{d_i}$ for $1 \leqslant i \leqslant u-1$. The correspondence $X \mapsto (x_i)$ gives an isomorphism of $\mathfrak{l}[\delta_G]$ onto $\prod_{1\leqslant i\leqslant u-1} \mathfrak{gl}(d_i, K) \times \mathfrak{so}(d_u, K)$ (or $\times \mathfrak{sp}(d_u/2, K)$ resp.). Under this isomorphism, a nilpotent class of $\mathfrak{l}[\delta_G]$ is parametrized by a system of Jordan types of x_i's: $(\beta_1, \beta_2, \cdots, \beta_{u-1}; \beta_u)$, where β_u, if $d_u \neq 0$, satisfies (BD1) or (C1) accordingly. Then we define $\mathrm{Ind}\,[\beta_1^\vee, \beta_2^\vee, \cdots, \beta_{u-1}^\vee; \beta_u^\vee]$ as the G_A-parabolic type for N given by $\mathrm{Ind}\,[\beta_1^\vee, \beta_2^\vee, \cdots, \beta_{u-1}^\vee, \beta_u^\vee, \beta_{u-1}^\vee, \cdots, \beta_2^\vee, \beta_1^\vee]$. The following is our second main theorem.

Theorem 2. *Let G be of type B_n, C_n or D_n, and $\delta_G = [d_1, d_2, \cdots, d_{u-1}; d_u]$ a G-parabolic type. Put $\mathfrak{p} = \mathfrak{p}[\delta_G]$, $\mathfrak{l} = \mathfrak{l}[\delta_G]$. For a nilpotent class C with Jordan type $(\beta_1, \beta_2, \cdots, \beta_{u-1}; \beta_u)$, let $C' = \mathrm{Ind}_{\mathfrak{l},\mathfrak{p}}^{\mathfrak{g}} C$ be its induced class. Then the Jordan type of C' is given* by $(T_G \circ \vee)\,\mathrm{Ind}\,[\beta_1^\vee, \beta_2^\vee, \cdots, \beta_{u-1}^\vee; \beta_u^\vee]$.

As a consequence of this theorem, we can characterize Richardson classes and fundamental classes (those which can not be induced from $\mathfrak{l} \neq \mathfrak{g}$).

Theorem 3. *Let G be of type B_n, C_n or D_n. Let C be a nilpotent class of $\mathfrak{g}$ with G-Jordan type $\alpha = (p_1, p_2, \cdots, p_s)$.* (i) *$C$ is a Richardson class corresponding to the parabolic subalgebra $\mathfrak{p}[\delta_G]$ with G-parabolic type $\delta_G = [d_1, d_2, \cdots, d_{u-1}; d_u]$ if and only if $\alpha = \delta^{\vee G}$ for $\delta = [d_1, d_2, \cdots, d_{u-1}, d_u, d_{u-1}, \cdots, d_1]$.* (ii) *$C$ is fundamental if and only if the multiplicities $r_j = \#\{i; p_i = j\}$ satisfy the following: if $r_j \neq 0$, then $r_i \neq 0$ for any $i < j$; and $r_j \neq 2$ for j odd* (resp. even) *when G is of type B or D* (resp. of type C).

We remark that the assertion (i) is essentially given in [4]. For the further study of Richardson classes, see [5] and also [9, §§4–5]. The condition on α in (ii) is exactly the condition for that α can not be expressed as $\beta^{\vee G}$ for a partition $\beta = [b_1, b_2, \cdots, b_t]$ of N with b_j not all different.

§4. Sketch of the proof of Theorem 2. We know that in $\mathrm{Jor}_G(\mathrm{Cl}(C'))$, there exists a unique maximal (with respect to $\succcurlyeq_G$) G-Jordan type α' which corresponds to C'. On the other hand, $C_A = G_A(C)$ is a G_A-conjugacy class. Consider G_A-parabolic type δ in Theorem 3 and G_A-induced class $C'_A = \mathrm{Ind}_{\mathfrak{l}[\delta],\mathfrak{p}[\delta]}^{\mathfrak{g}} C_A$. Then by Theorem 1, its G_A-Jordan type γ is given by $\gamma = [\mathrm{Ind}\,[\beta_1^\vee, \beta_2^\vee, \cdots, \beta_{u-1}^\vee; \beta_u^\vee]]^\vee$. Thus we see that $\mathrm{Jor}_G \cap \mathrm{Jor}_A(\gamma) \supset \mathrm{Jor}_G(\mathrm{Cl}(C'))$. By Lemma, the left hand side contains a unique maximal element $T_G\gamma$ with respect to $\succcurlyeq_A$. So we have $T_G\gamma \succcurlyeq_A \alpha'$.

To prove $T_G\gamma = \alpha'$, it is sufficient to find a nilpotent element Y with G-Jordan type $T_G\gamma$ from $C + \mathfrak{n}$, because $\mathrm{Cl}(C') \supset C + \mathfrak{n}$. We can choose Y by reducing the situation essentially to each simple step in the operation T_G: replacement of (q_k, q_m) by (q_k-1, q_m+1) (for the above γ, $m=k+1$ always), and then by checking each step explicitly. The detail is quite similar as the discussions in [9, §§ 4–5].

Note. I used in [5, § 4] Mizuno's result on the structure constants for type E_8 in [7, Table 12], in case of type E_7. So it would be better remarking here a complete correction to the Table 12: (1) add "+" at (12, 37) (misprint); (2) change signs at (I, J) and (J, I) for (I, J) = (32, 145), (107, 57), (38, 145), (44, 145), (112, 145), (120, 58).

References (continued from [I])

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