59. On the Killing Radical of Lie Triple Algebras

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Introduction. It is known that the radical of a finite dimensional Lie algebra $\mathfrak{A}$ over a field of characteristic zero is the orthogonal complement of the derived algebra $\mathfrak{A}^{(1)}=[\mathfrak{A},\mathfrak{A}]$ with respect to the Killing form α of $\mathfrak{A}$ (e.g. [1, § 5], [2, Ch. III–5]). As T. S. Ravisankar has pointed out in [12] and [13], O. Loos has shown, in the course of the proof of Satz 3 in [11], an analogous result for the radical (cf. [10]) $\mathfrak{r}$ of a Lie triple system $\mathfrak{g}$, that is, $\mathfrak{r}=\{X\in\mathfrak{g}\,|\,\beta(X,\mathfrak{g}^{(1)})=0\}$, where β is the Killing form of $\mathfrak{g}$ and $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g},\mathfrak{g}]$.

The purpose of this paper is to investigate solvability and semisimplicity of Lie triple algebras by introducing the concept of Killing radical (briefly K-radical) of a Lie triple algebra $\mathfrak{g}$ in an analogous way as the characterizations of radicals of Lie algebras and Lie triple systems mentioned above. Under a condition on $\mathfrak{g}$ we show that $\mathfrak{g}$ is K-solvable (resp. K-semisimple) if and only if its standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$ is a solvable (resp. semisimple) Lie algebra (Theorem 1).

1. Let $\mathfrak{g}$ be a finite dimensional Lie triple algebra (general Lie triple system in [15]) over a field of characteristic 0. Here, *Lie triple algebra* $\mathfrak{g}$ is an anti-commutative algebra with a trilinear operation $\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ denoted by $D(X,Y)Z$ for $X,Y,Z\in\mathfrak{g}$ satisfying the following conditions: (i) $D(X,X)Z=0$, (ii) $\mathfrak{S}\{(XY)Z+D(X,Y)Z\}=0$, (iii) $\mathfrak{S}D(XY,Z)W=0$, (iv) $D(X,Y)(ZW)=(D(X,Y)Z)W+Z(D(X,Y)W)$ and (v) $[D(X,Y),D(Z,W)]=D(D(X,Y)Z,W)+D(Z,D(X,Y)W)$, where $X,Y,Z,W\in\mathfrak{g}$ and $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. It should be noted that a Lie triple algebra $\mathfrak{g}$ is reduced to Lie algebra as an anti-commutative algebra if the trilinear multiplication $D(X,Y)Z$ vanishes identically, and that $\mathfrak{g}$ is reduced to Lie triple system under the ternary multiplication $[X,Y,Z]=D(X,Y)Z$ if it is a trivial algebra, i.e., $XY=0$ for $X,Y\in\mathfrak{g}$. The *standard enveloping Lie algebra* of $\mathfrak{g}$ is the Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$ under the bracket operation $[X,Y]=XY+D(X,Y)$, $[U,X]=-[X,U]=U(X)$, $[U,V]=UV-VU$ for $X,Y\in\mathfrak{g}$ and $U,V\in D(\mathfrak{g},\mathfrak{g})$, where $D(\mathfrak{g},\mathfrak{g})$ is the Lie algebra of all inner derivations $D(X,Y)\in\mathrm{End}\,(\mathfrak{g})$ for $X,Y\in\mathfrak{g}$. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if $\mathfrak{g}\mathfrak{h}\subset\mathfrak{h}$ and $D(\mathfrak{g},\mathfrak{h})\mathfrak{g}\subset\mathfrak{h}$ hold. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then $\mathfrak{B}=\mathfrak{h}\oplus D(\mathfrak{g},\mathfrak{h})$ is an ideal of the Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$. $\mathfrak{g}$ is *simple* if it has no nonzero proper ideal.

Let α denote the Killing form of the standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$ of $\mathfrak{g}$. We have introduced in [8] the concept of *Killing-Ricci form* β of $\mathfrak{g}$ as $\beta(X, Y)=\alpha(X, Y)$ for $X, Y \in \mathfrak{g}$, and shown that β is an invariant form on $\mathfrak{g}$ if the trilinear form

(1.1) $\gamma(X, Y, Z)=\mathrm{tr.}\, D(X, Y)L(Z), \qquad X, Y, Z \in \mathfrak{g},$

vanishes identically, where $L(Z)$ denotes the left multiplication by Z. The form γ can be written as

(1.2) $\gamma(X, Y, Z)=\alpha(D(X, Y), Z).$

If $\mathfrak{g}$ is reduced to Lie algebra, then β is the Killing form of the Lie algebra $\mathfrak{g}$. On the other hand, if $\mathfrak{g}$ is reduced to Lie triple system, then β is the Killing form of the Lie triple system in the sense of T.S. Ravisankar [13].

The following results have been shown in [8, Theorems 1, 2]:

Proposition 1. *Suppose that the trilinear form γ of $\mathfrak{g}$ vanishes identically. Then*;

(1) *The Killing-Ricci form β of $\mathfrak{g}$ is nondegenerate if and only if the standard enveloping Lie algebra $\mathfrak{A}$ is a semisimple Lie algebra.*

(2) *If β is nondegenerate, then $\mathfrak{g}$ is decomposed into a direct sum of mutually orthogonal simple ideals $\mathfrak{g}_i$'s with respect to β as*

(1.3) $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_r\,;\ \beta=\beta_1+\cdots+\beta_r,$

where each β_i is the Killing-Ricci form of $\mathfrak{g}_i$.

Remark 1. The result (2) in the above proposition is reduced to one of § 10 in [14] if $\mathfrak{g}$ is reduced to Lie triple system.

Remark 2. By using the direct sum decomposition (1.3) of Lie triple algebra we have obtained in [9] the decomposition of homogeneous systems (cf. [5], [6]) and of homogeneous Lie loops (cf. [3], [4]).

2. In the rest of this paper, we assume that the trilinear form γ given by (1.1) vanishes identically. Denote by $\mathfrak{g}^{(1)}=\mathfrak{g}\mathfrak{g}+D(\mathfrak{g},\mathfrak{g})\mathfrak{g}$ the ideal of $\mathfrak{g}$ generated by $\mathfrak{g}\mathfrak{g}$ and $D(\mathfrak{g},\mathfrak{g})\mathfrak{g}$. By the *Killing radical* (*K-radical*) of $\mathfrak{g}$ we mean the ideal $\mathfrak{r}_K=\{X \in \mathfrak{g} \mid \beta(X, \mathfrak{g}^{(1)})=0\}$. As mentioned in the introduction, the K-radical of $\mathfrak{g}$ is reduced to the radical of the Lie algebra (resp. Lie triple system) if $\mathfrak{g}$ is reduced to Lie algebra (resp. Lie triple system). The Lie triple algebra $\mathfrak{g}$ is *K-solvable* if $\mathfrak{r}_K=\mathfrak{g}$, and $\mathfrak{g}$ is *K-semisimple* if $\mathfrak{r}_K=\{0\}$. An ideal $\mathfrak{h}$ of $\mathfrak{g}$ is *K-solvable in* $\mathfrak{g}$ if $\beta(\mathfrak{h}, \mathfrak{g}^{(1)})=0$.

By using $\gamma=0$ and (1.2) we have;

Proposition 2. *An ideal $\mathfrak{h}$ of $\mathfrak{g}$ is K-solvable in $\mathfrak{g}$ if and only if the ideal $\mathfrak{B}=\mathfrak{h}\oplus D(\mathfrak{g},\mathfrak{h})$ of the Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$ is solvable.*

Now, we have the following;

Theorem 1. *Let $\mathfrak{g}$ be a finite dimensional Lie triple algebra over a field of characteristic zero, and assume that the trilinear form γ*

given by (1.1) *vanishes. Then*;

(1) $\mathfrak{g}$ *is K-solvable if and only if its standard enveloping Lie algebra* $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$ *is a solvable Lie algebra.*

(2) $\mathfrak{g}$ *is K-semisimple if and only if the Lie algebra* $\mathfrak{A}$ *is semisimple.*

Proof. (1) is an immediate consequence of Proposition 2. In [8, Theorem 1], we have proved that $\mathfrak{g}=\mathfrak{g}^{(1)}$ holds if the Killing-Ricci form β is nondegenerate. Hence, if $\mathfrak{A}$ is semisimple, then β is nondegenerate by Proposition 1 (1), and we get $\mathfrak{r}_K=(\mathfrak{g}^{(1)})^{\perp}=\mathfrak{g}^{\perp}=0$, that is, $\mathfrak{g}$ is K-semisimple. On the other hand, if $\mathfrak{A}$ is not semisimple, then β is degenerate and $\mathfrak{r}_K=(\mathfrak{g}^{(1)})^{\perp}\supset\mathfrak{g}^{\perp}\neq\{0\}$, that is, $\mathfrak{g}$ is not K-semisimple. Thus, (2) is shown.

Theorem 2. *Under the same assumptions as in Theorem* 1, $\mathfrak{g}$ *is K-semisimple if and only if it is decomposed into the direct sum*

$$\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_r \tag{2.1}$$

of simple and K-semisimple ideals $\mathfrak{g}_i$*'s* $(i=1,2,\cdots,r)$ *of dimension greater than* 1 *such that* $\beta=\beta_1+\cdots+\beta_r$, *where each* β_i *is the Killing-Ricci form of* $\mathfrak{g}_i$.

Proof. By (1) of Proposition 1 and (2) of Theorem 1, $\mathfrak{g}$ is K-semisimple if and only if its Killing-Ricci form β is nondegenerate. Hence, if $\mathfrak{g}$ is K-semisimple, then the decomposition (2.1) into simple ideals follows from (2) of Proposition 1. In this case, each β_i is nondegenerate Killing-Ricci form of $\mathfrak{g}_i$, so that $\mathfrak{g}_i$ is K-semisimple and $\dim\mathfrak{g}_i>1$. Conversely, suppose that $\mathfrak{g}$ is decomposed into (2.1) with simple and K-semisimple ideals $\mathfrak{g}_i$. Then, β is nondegenerate since $X=X_1+\cdots+X_r$, $X_i\in\mathfrak{g}_i$, satisfies $\beta(X,\mathfrak{g})=0$ if and only if $\beta_i(X_i,g_i)=0$, $i=1,2,\cdots,r$, and since $\mathfrak{g}_i$ is K-semisimple.

3. In our paper [7], we have treated a kind of solvability of Lie triple algebras given as follows: For any ideal $\mathfrak{h}$ of $\mathfrak{g}$, set $\mathfrak{h}^{(0)}=\mathfrak{h}$, $\mathfrak{h}^{(1)}=\mathfrak{h}\mathfrak{h}+D(\mathfrak{g},\mathfrak{h})\mathfrak{h}$ and $\mathfrak{h}^{(i+1)}=\mathfrak{h}^{(i)}\mathfrak{h}^{(i)}+D(\mathfrak{h},\mathfrak{h})\mathfrak{h}^{(i)}+D(\mathfrak{g},\mathfrak{h}^{(i)})\mathfrak{h}^{(i)}$ for $i\geqq 1$. Then, in the chain $\mathfrak{h}=\mathfrak{h}^{(0)}\supset\mathfrak{h}^{(1)}\supset\cdots\supset\mathfrak{h}^{(i)}\supset\mathfrak{h}^{(i+1)}\supset\cdots$ of Lie triple subalgebras of $\mathfrak{g}$, each $\mathfrak{h}^{(i+1)}$ is an ideal of $\mathfrak{h}^{(i)}$ and $\mathfrak{h}^{(i)}/\mathfrak{h}^{(i+1)}$ is abelian (cf. [7, Proposition 1]). An ideal $\mathfrak{h}$ is *solvable in* $\mathfrak{g}$ if $\mathfrak{h}^{(i)}=\{0\}$ for some i. The *radical* of $\mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$. $\mathfrak{g}$ is *semisimple* if the radical of $\mathfrak{g}$ is zero.

Proposition 2 and Theorem 1 combined with Proposition 2, Theorems 1 and 3 in [7] imply the following;

Theorem 3. *Assume that the trilinear form* γ *of* $\mathfrak{g}$ *vanishes. Then*; (1) *If an ideal* $\mathfrak{h}$ *is solvable in* $\mathfrak{g}$, *then it is K-solvable in* $\mathfrak{g}$.

(2) *If* $\mathfrak{g}$ *is solvable Lie triple algebra, then it is K-solvable.*

(3) *If* $\mathfrak{g}$ *is K-semisimple, then it is semisimple.*

Remark 3. It is not known whether K-solvability coincides with solvability in [7] or not.

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