109. *Lerch's Theorem for Analytic Functional*

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§0. **Introduction.** In 1950's, J. Mikusiński established "Operational Calculus". In his theory, Titshmarsh's convolution theorem plays a very important role. A large number of proofs of that theorem have been known. Mikusiński's proof makes use of Lerch's theorem. (See [2].) In this article, we will prove Lerch's theorem by means of the theory of analytic functionals. Our main result is as follows:

Theorem. *Let T be an analytic functional on C^n carried by the direct product $K=\prod_{i=1}^n([a_i, b_i]+\sqrt{-1}[-\varepsilon_i, \varepsilon_i])$. Suppose that T satisfies the following condition:*

$$\limsup_{m_1+\cdots+m_n\to\infty}\left\{\exp\left(-\sum_{i=1}^n m_i a_i\right)|\tilde{T}(m_1, \cdots, m_n)|\right\}^{1/(m_1+\cdots+m_n)}\leqq 1,$$

where $\tilde{T}(z)=\langle T_\zeta, \exp(\zeta z)\rangle$ denotes the Fourier-Borel transformation of T. Then T is carried by the set $(a_1, \cdots, a_n)+\sqrt{-1}\prod_{i=1}^n[-\varepsilon_i, \varepsilon_i]$.

The following corollaries are direct consequences of Theorem. In the case of $n=1$, Corollary 2 coincides with Lerch's theorem.

Corollary 1. *Let T be a hyperfunction on R^n whose support is contained in the direct product of closed intervals $K=\prod_{i=1}^n[a_i, b_i]$. Suppose that T satisfies the same condition as in Theorem. Then the support of T concentrates to the point $(a_1, \cdots, a_n)$.*

Corollary 2. *Let T be a complex valued continuous function on the direct product of closed intervals $K=\prod_{i=1}^n[a_i, b_i]$, $0\leqslant a_i\leqslant b_i$. Suppose that $T(t)$ satisfies the following condition:*

$$\left|\int_K \exp\left(\sum_{i=1}^n m_i t_i\right)T(t_1, \cdots, t_n)dt_1\cdots dt_n\right|\leqq M \quad \text{for any } (m_1, \cdots, m_n)\in N^n.$$

Then $T(t)$ vanishes identically.

§1. **Avanissian-Gay transformation of analytic functional.** In this section, we recall the definition of the Avanissian-Gay transformation of an analytic functional carried by a convex compact set. According to Avanissian-Gay [1], we will define the Avanissian-Gay transformation of an analytic functional T carried by a convex compact set K as follows:

$$G_K(T)(w)=\left\langle T_\zeta, \prod_{i=1}^n(1-w_i\exp(\zeta_i))^{-1}\right\rangle.$$

The Avanissian-Gay transformation $G_K(T)$ has the following properties (see [1]):

(1) $G_K(T)(w)$ is holomorphic in $\prod_{i=1}^n (\boldsymbol{C}\backslash\exp(-\boldsymbol{K}_i))$, where $\boldsymbol{K}_i$ is the i-th projection of K.

(2) The Taylor expansion $G_K(T)(w)=\sum_{m\in\boldsymbol{N}^n}\tilde{T}(m)w^m$ is valid in a neighbourhood of 0, where $\tilde{T}(z)$ denotes the Fourier-Borel transformation of T.

(3) $\lim_{|w|\to\infty} G_K(T)(w)=0$.

§2. **Proof of Theorem.** First of all, we assume that ε_i (>0) is less than π without loss of generality. We consider the Taylor expansion of $G_K(T)(w)$ at $w=0$. By the assumption of Theorem, the Taylor expansion of $G_K(T)(w)$ converges in the polydisc $\prod_{i=1}^n \boldsymbol{D}_i$, where $\boldsymbol{D}_i=\{w\,;\,|w|<\exp(-a_i)\}$. Accordingly, $G_K(T)(w)$ is holomorphic in the polydisc $\prod_{i=1}^n \boldsymbol{D}_i$. We want to show that $G_K(T)(w)$ is holomorphic in the region $\prod_{i=1}^n (\boldsymbol{C}\backslash\exp(-\boldsymbol{L}_i))$, where $\boldsymbol{L}_i=a_i+\sqrt{-1}[-\varepsilon_i,\varepsilon_i]$. For the simplicity, we will confine ourselves to the 2-dimensional case. Let us consider the following integral:

$$G_\Gamma(w)=-(2\pi i)^{-1}\int_\Gamma G_K(T)(\zeta_1,w_2)(\zeta_1-w_1)^{-1}d\zeta_1,$$

where Γ is a contour which surrounds compact set $\exp(-\boldsymbol{K}_1)$, $\boldsymbol{K}_i=[a_i,b_i]+\sqrt{-1}[-\varepsilon_i,\varepsilon_i]$, $i=1,2$. By Property (1) of $G_K(T)(w)$, $G_\Gamma(w)$ is holomorphic in the region $(\boldsymbol{C}\backslash\Gamma)\times(\boldsymbol{C}\backslash\exp(-\boldsymbol{K}_2))$. From Property (3) and Cauchy's integral theorem, $G_\Gamma(w)$ equals to $G_K(T)(w)$ in the region (the exterior of $\Gamma)\times\boldsymbol{D}_2$. Now we fix w_2 in D_2. In this case, $G_K(T)(w)$ is holomorphic in $\boldsymbol{D}_1$ with respect to variable w_1. Accordingly, we can deform the contour Γ in $\boldsymbol{D}_i$ by means of Cauchy's integral theorem in such a way that $G_K(T)(w)$ has an analytic continuation to the region $(\boldsymbol{C}\backslash\exp(-\boldsymbol{L}_1))\times D_2$. Hence $G_K(T)(w)$ has a single valued analytic continuation to the region $\prod_{i=1}^2 (\boldsymbol{C}\backslash\exp(-\boldsymbol{K}_i))\cup(\boldsymbol{C}\backslash\exp(-\boldsymbol{L}_1))\times\boldsymbol{D}_2$. We can prove similarly that $G_K(T)(w)$ has an analytic continuation to the region $\boldsymbol{D}_1\times(\boldsymbol{C}\backslash\exp(-\boldsymbol{L}_2))$. Therefore $G_K(T)(w)$ can be continued analytically to the region $\prod_{i=1}^2 (\boldsymbol{C}\backslash\exp(-\boldsymbol{K}_i))\cup\prod_{i=1}^2 \boldsymbol{D}_i\cup(\boldsymbol{C}\backslash\exp(-\boldsymbol{L}_1))\times\boldsymbol{D}_2\cup\boldsymbol{D}_1\times(\boldsymbol{C}\backslash\exp(-\boldsymbol{L}_2))$, which is equal to $\prod_{i=1}^2 (\boldsymbol{C}\backslash\exp(-\boldsymbol{L}_i))$. $G(w)$ denotes the analytic continuation of $G_K(T)(w)$. Now we define an analytic functional $\check{T}$ as follows:

$$\langle\check{T},h\rangle=(2\pi i)^{-2}\int_{\Gamma_1\times\Gamma_2} G(w)h(-\log w)w^{-1}dw,$$

where $h(z)$ is a holomorphic test function defined on a neighbourhood of $\prod_{i=1}^2 L_i$, and Γ_i is a contour which surrounds the interval $\boldsymbol{L}_i$. According to Avanissian-Gay [1], $\check{T}$ is carried by $\prod_{i=1}^2 L_i$. Also, $\check{T}$ is an extension of the analytic functional T. If we put $\varepsilon_i=0$, then we obtain Corollary 1. For the details of the relation between analytic functionals and hyperfunctions, the reader is asked to refer, for example, to Morimoto [3]. Corollary 2 is a direct consequence of Corollary 1.

References

[1] V. Avanissian and R. Gay: Sur une transformation des fonctionnelles analytiques et ses applications aux fonctions entières de plusieurs variables. Bull. Soc. Math., **103**, 341–384 (1975).
[2] J. Mikusiński: Operational Calculus. Pergamon Press, London (1959).
[3] M. Morimoto: Introduction to Sato's Hyperfunction Theory. Kyoritsu Press, Tokyo (1976) (in Japanese).