121. On Local Isometric Immersions of Riemannian Symmetric Spaces

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1. The present note is a brief summary of our forthcoming paper [2] concerning local isometric or conformal immersions of Riemannian symmetric spaces into the Euclidean spaces. (We assume the differentiability of class C^∞.)

Let $M=G/K$ be an n-dimensional Riemannian symmetric space and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of G and K, respectively. We denote by $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ the canonical decomposition of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. Then the curvature transformation $R(X, Y): \mathfrak{m} \to \mathfrak{m}$ $(X, Y \in \mathfrak{m})$ at the origin is given by $R(X, Y)Z=-[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$ (see [6]). We define a non-negative integer $c(G/K)$ by

$$c(G/K)=1/2\cdot \max_{X,Y\in\mathfrak{m}} \operatorname{rank} R(X, Y).$$

We remark that $c(G/K)$ is determined by the infinitesimal property of G/K. Then our first result is

Proposition 1 (cf. Matsumoto [7]). *Any open Riemannian submanifold of an n-dimensional Riemannian symmetric space $M=G/K$ cannot be isometrically immersed in $\boldsymbol{R}^{n+c(M)-1}$.*

Proof. Suppose that there exists an isometric immersion f of an open Riemannian submanifold U of M into $\boldsymbol{R}^{n+c(M)-1}$. Let α be the second fundamental form of f and $T_x^\perp U$ be the normal space to U at $x\in U$. For each $\xi \in T_x^\perp U$ we define a symmetric endomorphism A_ξ of T_xM by $g(A_\xi(X), Y)=\langle\alpha(X, Y), \xi\rangle$ $(X, Y \in T_xM)$ where g is the Riemannian metric of M and $\langle\ ,\ \rangle$ is the inner product of $\boldsymbol{R}^{n+c(M)-1}$. Then the Gauss equation at x can be written in the form $R(X, Y)Z=A_{\alpha(Y,Z)}X-A_{\alpha(X,Z)}Y$ for $X, Y, Z \in T_xM$ (cf. [6]). Hence for $X, Y \in T_xM$ we have

$$\dim\{R(X, Y)Z \mid Z \in T_xM\} \leqq \dim\{A_{\alpha(Y,Z)}X \mid Z \in T_xM\} + \dim\{A_{\alpha(X,Z)}Y \mid Z \in T_xM\} \leqq 2\dim T_x^\perp U = 2c(G/K)-2,$$

which contradicts

$$2c(G/K)=\max_{X,Y\in T_xM} \dim\{R(X, Y)Z \mid Z\in T_xM\}. \qquad \text{Q.E.D.}$$

Remark. A similar result holds for general Riemannian manifolds (M, g). For details, see [2].

Combining the result of J. D. Moore [8] with Proposition 1, we can prove

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Proposition 2. *Any open Riemannian submanifold of $M=G/K$ cannot be conformally immersed in $\boldsymbol{R}^{n+c(M)-3}$.*

The following lemma is easy to verify.

Lemma 3. (1) *Let $M=M_1\times\cdots\times M_p$ be a product of Riemannian symmetric spaces. Then $c(M)=\sum_{i=1}^{p} c(M_i)$.*

(2) *Let M be a Riemannian symmetric space of compact type and M^* be its non-compact dual space. Then $c(M)=c(M^*)$.*

2. We determine the integers $c(G/K)$ for simply connected irreducible Riemannian symmetric spaces of compact type. Then by Lemma 3 and the fact $c(\boldsymbol{R}^n)=0$, we know the value $c(G/K)$ for all Riemannian symmetric spaces. Now our main result is

Theorem 4. *Let $M=G/K$ be a simply connected irreducible Riemannian symmetric space of compact type. If G/K is not isomorphic to any real Grassmann manifold, then*

$$c(G/K)=1/2\cdot(\dim G/K-\operatorname{rank} G+\operatorname{rank} K).$$

For $G/K=SO(p+q)/SO(p)\times SO(q)$ $(p\geqq q\geqq 1)$,

$$c(G/K)=\begin{cases}[pq/2], & \text{if } q=\text{even or } 2q\geqq p\geqq q,\ q=\text{odd},\\ p(q-1)/2+q, & \text{if } p>2q \text{ and } q=\text{odd},\end{cases}$$

where [ ] is the Gauss symbol.

Remark. (1) If M is not isomorphic to $SO(p+q)/SO(p)\times SO(q)$ $(p>2q+1$ and $q=\text{odd})$, then $c(G/K)=1/2\cdot(\dim G/K-\operatorname{rank} G+\operatorname{rank} K)$. Hence for most of the Riemannian symmetric spaces M, we have $c(M)\sim 1/2\cdot\dim M$ and therefore by Propositions 1 and 2 they cannot be isometrically or conformally immersed into the Euclidean spaces in codimension $\sim 1/2\cdot\dim M$. Note that many Riemannian symmetric spaces of compact type M can be globally isometrically embedded into the Euclidean spaces in codimension $\sim\dim M$ (see [5]).

(2) In [4] and [3] Heitsch, Lawson and Donnelly proved that the Riemannian symmetric spaces $SO(2m+1)$, $U(2m+1)$ and $SU(2m+1)/SO(2m+1)$ cannot be globally conformally immersed into the Euclidean spaces in codimension $2m-1$ by calculating the Chern-Simons invariants. But our estimates obtained above are better than theirs for large m because for these spaces the integers $c(M)$ are quadratic polynomials of m and hence $c(M)\gg 2m-1$ for large m.

(3) In general our results are not best possible. For example the spaces of negative constant curvature of dimension n $(\geqq 2)$ cannot be isometrically immersed in $\boldsymbol{R}^{2n-2}$ even locally [9]. But by Lemma 3 and Theorem 4 the value $c(M)$ is 1 in this case and hence our result is not best possible for $n\geqq 3$. For other examples, see [1] and [2].

For the proof of Theorem 4, see our forthcoming paper [2].

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