8. *Some New Linear Relations for Odd Degree Polynomial Splines at Mid-Points*

By Manabu SAKAI
Department of Mathematics, Faculty of Science, Kagoshima University

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By making use of the B-spline $Q_{p+1}(x)$:

$$Q_{p+1}(x)=(1/p\,!)\sum_{i=0}^{p+1}(-1)^i\binom{p+1}{i}(x-i)_+^p,$$

we consider the spline function $s(x)$ of the form

$$s(x)=\sum_{i=-p}^{n}\alpha_i Q_{p+1}\left(\frac{x}{h}-i+\frac{1}{2}\right),\qquad nh=1$$

where

$$(x-i)_+^p=\begin{cases}(x-i)^p & \text{for } x\geq i\\ 0 & \text{for } x<i.\end{cases}$$

It is well known that

(i) s is a polynomial of degree p on $\left[\left(i-\frac{1}{2}\right)h,\ \left(i+\frac{1}{2}\right)h\right]$,

(ii) $s\in C^{p-1}(-\infty,\infty)$.

Let p and k be integers such that $1\leq k\leq p-1$, then the following consistency relation holds

$$h^{-k}\left\{Q_{p+1}^{(k)}\left(p+1-\frac{1}{2}\right)s_i+Q_{p+1}^{(k)}\left(p-\frac{1}{2}\right)s_{i+1}+\cdots+Q_{p+1}^{(k)}\left(\frac{1}{2}\right)s_{i+p}\right\}$$

$$(*)\qquad =Q_{p+1}\left(p+1-\frac{1}{2}\right)s_i^{(k)}+Q_{p+1}\left(p-\frac{1}{2}\right)s_{i+1}^{(k)}+\cdots+Q_{p+1}\left(\frac{1}{2}\right)s_{i+p}^{(k)}\quad([2]).$$

Here $s_i=s(ih)$ and $s_i^{(k)}=s^{(k)}(ih)$.

From now on, let p and k be odd and even integers, respectively. Since k is even, in virtue of the properties:

$$Q_{p+1}(x)\equiv Q_{p+1}(p+1-x)$$
$$Q_{p+1}(x)\equiv 0\qquad\text{for } x\leq 0,\ x\geq p+1,$$

we have

$$c_j^{(l)}=Q_{p+1}^{(l)}\left(p+\frac{1}{2}-j\right)-Q_{p+1}^{(l)}\left(p+\frac{3}{2}-j\right)+\cdots$$
$$=(-1)^{j-p}\left\{Q_{p+1}^{(l)}\left(p+\frac{1}{2}\right)-Q_{p+1}^{(l)}\left(p-\frac{1}{2}\right)+\cdots\right\}$$
$$\text{for } l=0,\ k \text{ and } j=p,\ p+1,\ \cdots.$$

Since p is odd, in virtue of the property:

$$Q_{p+1}^{(l)}\left(p+\frac{1}{2}-j\right)=Q_{p+1}^{(l)}\left(j+\frac{1}{2}\right)$$
$$\text{for } l=0,\ k \text{ and } j=p,\ p+1,\ \cdots,$$

we have

$$c_j^{(0)}=0 \quad \text{and} \quad c_j^{(k)}=0 \qquad \text{for } j=p,\ p+1,\ \cdots.$$

Hence, an alternating sum obtained by writing down equation $(*)$, substracting equation $(*)$ with i replaced by $i+1$, adding equation $(*)$ with i replaced by $i+2$ and so on is equal to the short term consistency relation between s_j and $s_j^{(k)}$, $j=i,\ i+1,\ \cdots,\ i+p-1$:

$$h^{-k}\{c_0^{(k)}s_i+c_1^{(k)}s_{i+1}+\cdots+c_{p-1}^{(k)}s_{i+p-1}\}=c_0^{(0)}s_i^{(k)}+c_1^{(0)}s_{i+1}^{(k)}+\cdots+c_{p-1}^{(0)}s_{i+p-1}^{(k)}$$

for odd p and even k such that $2\leq k\leq p-1$.

Since $Q_{p+1}^{(l)}\left(p+\dfrac{1}{2}\right)-Q_{p+1}^{(l)}\left(p-\dfrac{1}{2}\right)+\cdots-Q_{p+1}^{(l)}\left(\dfrac{1}{2}\right)=0$

$$l=0,\ k,$$

we have

$$c_j^{(0)}=c_{p-1-j}^{(0)} \quad \text{and} \quad c_j^{(k)}=c_{p-1-j}^{(k)},\ j=0,\ 1,\ \cdots,\ p-1.$$

As examples of the above relations, let $s(x)$ be cubic and quintic splines, respectively. Then we have

(i) cubic spline,

$$\frac{1}{2}h^{-2}(s_i-2s_{i+1}+s_{i+2})=(1/48)\,(s_i''+22s_{i+1}''+s_{i+2}'');$$

(ii) quintic spline,

$$\frac{1}{2}h^{-4}(s_i-4s_{i+1}+6s_{i+2}-4s_{i+3}+s_{i+4})$$
$$=(1/3840)(s_i^{(4)}+236s_{i+1}^{(4)}+1446s_{i+2}^{(4)}+236s_{i+3}^{(4)}+s_{i+4}^{(4)}),$$
$$(1/48)h^{-2}(s_i+20s_{i+1}-42s_{i+2}+20s_{i+3}+s_{i+4})$$
$$=(1/3840)(s_i''+236s_{i+1}''+1446s_{i+2}''+236s_{i+3}''+s_{i+4}'').$$

These short term consistency relations are useful for the investigation of the spline interpolation at mid-points and the application of splines to the numerical solution of a boundary value problem.

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References

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