17. *Formation of Singularities for Hamilton-Jacobi Equation. I*

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§1. **Introduction.** This note is concerned with the singularities of global solution of Hamilton-Jacobi equation in two space dimensions:

$$(1)\qquad \frac{\partial u}{\partial t}+f\left(\frac{\partial u}{\partial x}\right)=0 \qquad \text{in } \{t>0,\ x\in R^2\},$$

$$(2)\qquad u(0,x)=\varphi(x)\in C_0^\infty(R^2),$$

where $C_0^\infty(R^2)$ is a set of C^∞-functions whose supports are compact. In this note we assume that f is C^∞ and uniformly convex. It's well known that, even for smooth initial data, the Cauchy problem (1) and (2) doesn't admit a smooth solution for all t. Therefore we consider a generalized solution of (1), (2) whose definition will be given in §2. The existence of global generalized solutions is already established by many authors. (See [1] and its references.)

For a single conservation law in one space dimension, a solution satisfying the entropy condition is piecewise smooth for any smooth initial data in $\mathscr{S}=\{$rapidly decreasing functions$\}$ except for a subset of the first category ([3]–[5] and [8]). T. Debeneix [2] treated certain systems of conservation law which is essentially equivalent to Hamilton-Jacobi equation (1) in R^n ($n\leqslant 4$), and generalized the results of [8] to this case by the same method as [8].

One of the classical methods for solving first order non-linear equations is the characteristic one. Its weak point is that it's the local theory. The reason is due to the fact that a smooth mapping can't uniquely have the inverse at a point where its jacobian vanishes, i.e., that its inverse becomes many-valued there. Therefore the solution takes also many values in a neighborhood of critical points of a mapping H_t defined in §3. The aim of this note is to show how to choose up the reasonable value of its many values so that the solution is one-valued and continuous.

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§2. **Generalized solutions.** Let's put $p=(p_1,p_2)\in R^2$, and write

$$f'(p)=\left(\frac{\partial f}{\partial p_1}(p),\ \frac{\partial f}{\partial p_2}(p)\right) \quad\text{and}\quad f''(p)=\left[\frac{\partial^2 f}{\partial p_i \partial p_j}(p)\right]_{1\leqslant i,\, j\leqslant 2}.$$

In this note we assume that $f(p)$ is uniformly convex, i.e., $f''(p) \geqslant C > 0$ where C is constant. We now define a generalized solution of (1) and (2).

Definition. A Lipschitz continuous function $u(t,x)$ defined on $R^1 \times R^2$ is called a generalized solution of (1) and (2) if

i) $u(t,x)$ satisfies (1) almost everywhere in $R^1 \times R^2$ and (2) on $t=0$,

ii) $u(t,x)$ is semi-concave, i.e., there exists a constant $k>0$ such that

(3) $$u(t,x+y)+u(t,x-y)-2u(t,x) \leqslant k|y|^2$$

for any $x, y \in R^2$ and $t>0$.

It's well known that the Cauchy problem (1) and (2) has a unique global solution with the above properties i) and ii) ([6], [1]).

§ 3. Construction of solutions. The characteristic lines corresponding to the Cauchy problem (1) and (2) are determined by following equations:

$$\dot{x}_i = \partial f / \partial p_i(p), \quad \dot{p}_i = 0 \quad (i=1,2)$$

with the initial conditions

$$x_i(0) = y_i, \quad p_i(0) = \partial\varphi / \partial y_i(y) \quad (i=1,2).$$

On the characteristic line, the value $v(t,y)$ of the solution satisfies an equation

$$\dot{v} = -f(p) + \langle p, f'(p) \rangle, \qquad v(0) = \varphi(y),$$

where $\langle p, q \rangle$ means a scalar product of vectors p and q. Solving these equations, we have

(4) $$x = y + tf'(\varphi'(y)) \underset{\text{def.}}{=} H_t(y),$$

(5) $$v = \varphi(y) + t\{-f(\varphi'(y)) + \langle \varphi'(y), f'(\varphi'(y)) \rangle\}.$$

Then H_t is a smooth mapping from R_y^2 to R_x^2, and its jacobian is given by

$$\partial x / \partial y(t,y) = \det(I + tf''(\varphi'(y))\varphi''(y)).$$

Let's write $A(y) = f''(\varphi'(y))\varphi''(y)$ and $\lambda_1(y) \leqslant \lambda_2(y)$ be eigenvalues of $A(y)$. Assume $\min \lambda_1(y) = \lambda_1(y_0) = -M < 0$ and put $t_0 = 1/M$. Then, since $\partial x / \partial y\ (t,y) \neq 0$ for any $t < t_0$ and $y \in R^2$, we can uniquely solve (4) with respect to y and denote it by $y = y(t,x)$. We see $u(t,x) = v(t, y(t,x))$ is the smooth solution of (1) and (2) for $t < t_0$. Our problem is to construct the solution for $t > t_0$.

Suppose that $t - t_0$ is positive and sufficiently small. The jacobian of H_t vanishes on $\Sigma_t = \{y \in R^2;\ 1 + t\lambda_1(y) = 0\}$. Assume the condition

(A.1) $\lambda_1(y) \in C^2$, $\mathrm{grad}_y\, \lambda_1(y) \neq 0$ *on* Σ_t, *and* Σ_t *is a simple closed curve.*

In this case, Σ_t is parametrized as $\Sigma_t = \{(y_1(s), y_2(s));\ s \in I\}$ where I is an interval and $y_i(s) \in C^2(I)$ $(i=1,2)$. Put

$$\Sigma_t^e = \left\{ y(s_0) \in \Sigma_t;\ \frac{d}{ds} v(t, y(s)) = 0 \text{ at } s = s_0 \right\}.$$

By the definition of Whitney [10], a point Y in $\Sigma_t - \Sigma_t^e$ is a fold point of the mapping H_t, i.e.,

$$(d/ds)x(t, y(s)) \neq 0 \qquad \text{at } y=Y.$$

Lemma 1. *Suppose $Y \in \Sigma_t^e$. If a number of elements of Σ_t^e is two or $\partial v/\partial y \neq 0$ at $y=Y$, then it follows*

$$\frac{d}{ds}x(t, y(s)) = (I+tA(y))\frac{dy}{ds} = 0 \qquad \text{at } y=Y.$$

Assume here the following condition:

(A.2) *$\Sigma_t^e = \{Y_1, Y_2\}$ and Y_i $(i=1,2)$ are cusp points of H_t*, i.e.,

$$\frac{d^2}{ds^2}x(t, y(s)) \neq 0 \qquad \text{at } y=Y_i \ (i=1,2).$$

When we denote the restriction of $v(t, y)$ on Σ_t by $v_\Sigma(t, y)$, the function v_Σ takes its extremums on Σ_t^e. Especially, if we put $v(t, Y_i) = c_i$ $(i=1,2)$ and suppose $c_1 < c_2$, v_Σ takes its minimum at $y=Y_1$ and maximum at $y=Y_2$. Denote by D_t the interior of the curve Σ_t and by Ω_t the interior of $H_t(\Sigma_t)$. Then the curve $v(t, y)=c_i$ is tangent to D_t at $y=Y_i$ $(i=1,2)$. Moreover, when we put $H_t(Y_i)=X_i$ $(i=1,2)$, the curve $H_t(\Sigma_t)$ has the cusps at $x=X_1$ and X_2. See Fig. 1. When we solve the equation (4) with respect to y for any $x \in \Omega_t$, the solution $y=y(t, x)$ becomes three-valued. Write these values by $g_1(t, x)$, $g_2(t, x)$ and $g_3(t, x)$ where $g_2(t, x)$ is in D_t for any $x \in \Omega_t$. Then the solution $u(t, x) = v(t, y(t, x))$ also is three-valued on Ω_t, i.e., when one puts $u_i(t, x) = v(t, g_i(t, x))$ $(i=1,2,3)$, the solution takes the values $u_1(t, x)$, $u_2(t, x)$ and $u_3(t, x)$ on Ω_t. Next, pick up any number $c \in (c_1, c_2)$, and consider the image of the curve $\{y \in R^2; v(t, y)=c\}$ by H_t. Then its image intersects itself only at one point in Ω_t (see Fig. 1).

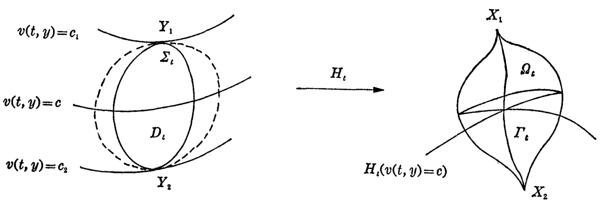

Fig. 1

Using these facts, we obtain the following

Lemma 2. i) *$u_1(t, x) < u_2(t, x)$ and $u_3(t, x) < u_2(t, x)$ for any $x \in \Omega_t$,* ii) *The set $\Gamma_t = \{x \in \Omega_t; u_1(t, x) = u_3(t, x)\}$ determines a smooth curve combining the points X_1 and X_2.*

The dotted curve in Fig. 1 means $H_t^{-1}(\Gamma_t)$. For the proofs of these properties, it's necessary to see the behavior of the inverse mapping H_t^{-1} in a neighborhood of Ω_t. With respect to this problem, we use the famous results of Whitney [10]. Since we are looking for a continuous and one-valued solution, we define the solution

$$(6)\qquad u(t,x)=\begin{cases} u_1(t,x) & \text{in } \Omega_{t,+}\equiv\{x\in\Omega_t;\ u_3(t,x)-u_1(t,x)>0\},\\ u_3(t,x) & \text{in } \Omega_{t,-}\equiv\{x\in\Omega_t;\ u_3(t,x)-u_1(t,x)<0\}.\end{cases}$$

§4. Semi-concavity of the solution $u(t,x)$. Let's $\vec{n}$ be a normal of Γ_t advancing from $\Omega_{t,-}$ to $\Omega_{t,+}$, and denote

$$\partial u/\partial x(t,x\pm 0)\equiv\lim_{\varepsilon\to+0}\partial u/\partial x(t,x\pm\varepsilon\vec{n})\qquad\text{for } x\in\Gamma_t.$$

Then the semi-concavity property (3) is equivalent to the following inequality;

$$(7)\qquad \langle\partial u/\partial x(t,x+0)-\partial u/\partial x(t,x-0),\vec{n}\rangle\leqslant 0\qquad\text{for } x\in\Gamma_t,$$

which is the entropy condition for a system of conservation law obtained by letting $\partial u/\partial x=w$ be unknown functions. On the other hand, as $\vec{n}$ advances from $\Omega_{t,-}$ to $\Omega_{t,+}$, it follows

$$(d/ds)(u_3(t,x+s\vec{n})-u_1(t,x+s\vec{n}))|_{s=0}\geqslant 0\qquad\text{for } x\in\Gamma_t,$$

which means that, when we write $\vec{n}=k(\partial u_3/\partial x-\partial u_1/\partial x)|_{\Gamma_t}$, k must be non-negative. Therefore (7) is easily obtained. It's already known that a Lipschitz-continuous and semi-concave solution is unique. Hence the solution constructed above is the reasonable one of (1) and (2).

§5. Collision of singularities. Let's Γ_1 and Γ_2 be singularities constructed as above. We use notations of Fig. 2. A collision of type (i) doesn't appear. In the case (ii), the solution becomes two-valued on a domain bounded by Γ_1 and Γ_2. By the similar discussion as in §3, we can uniquely pick up a one-valued continuous solution there. Its new singularity is written by a dotted curve in (ii) of Fig. 2. There is no problem for the case (iii).

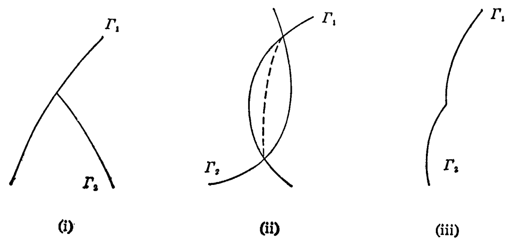

Fig. 2

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