43. Pluricanonical Mappings of Canonically Polarized Varieties

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In this note, we shall prove the following result.

Theorem. *Let V be a canonically polarized variety of dimension n over $\boldsymbol{C}$. Then there exists an integer N which depends only on n such that the m-th canonical mappings Φ_m of V are birational for all $m \geqq N$.*

Here, V is said to be canonically polarized, if it is non-singular, complete and if the canonical divisor $K(V)$ is ample.

To prove this, we need the following lemmas.

Lemma 1 (Matsusaka). *Let V be a canonically polarized variety of dimension n and let γ be $K(V)^{(n)}$. If $P_m(V) \geqq \gamma m^{n-1}+n$, then the m-th canonical mapping Φ_m is generically finite.*

Lemma 2 (Wilson). *Let V be a non-singular variety of dimension n. If there exists m such that the m-th canonical mapping Φ_m is generically finite and $P_m(V) \geqq n+2$ then Φ_{nm+1} is birational.*

Lemma 3. *Let V be a complete non-singular variety of dimension n over a field of characteristic zero. Assume that the m_b-th canonical mapping is birational. Then the m-th canonical mapping is birational for all $m \geqq \operatorname{Max}\{1, nm_b(m_b-1)\}$.*

Proof. Put $W_m=\Phi_m(V)$. Clearly, $\operatorname{Rat}(W_{km_b})=\operatorname{Rat}(W_{m_b})=\operatorname{Rat}(V)$ for all integers $k \geqq 1$. By Wilson's Lemma $\operatorname{Rat}(W_{nm_b+1})=\operatorname{Rat}(V)$. It suffices to show that we can find integers $\alpha, \beta \geqq 0$ such that $m=\alpha(nm_b+1)+\beta m_b$. In fact, we can find integers $q \geqq 1$, $nm_b(m_b-1)>r \geqq 0$ such that $m=qnm_b(m_b-1)+r$. Also, $r=sm_b+\alpha$ for $s \geqq 0$, $m_b>\alpha \geqq 0$. Hence $m-\alpha(nm_b+1)=\beta m_b$, where $\beta=n(q(m_b-1)-\alpha)+s$. Note that $\beta \geqq 0$.

Proof of Theorem. Since $K(V)$ is ample, it follows that $P_m(V)=\chi(V, \mathcal{O}(mK))=\Sigma_{i=0}^n(-1)^i \dim H^i(V, \mathcal{O}(mK))$ for $m \geqq 2$. Note that the leading coefficient of polynomial $\chi(V, \mathcal{O}(mK))$ is equal to $\gamma/n!$ Moreover if $P_k(V)>\gamma k^{n-1}+n-1$ (Matsusaka inequality) for one of value k such that $2 \leqq k \leqq n+2$, then we can find such a number N that all the m-th canonical mappings are birational for all $m \geqq N$, by virtue of Lemmas 1, 2 and 3.

Case 1. Assume $\gamma \leqq n-1$. If $P_m(V)>(n-1)(m^{n-1}+1)$ for one value m such that $2 \leqq m \leqq n+2$, then Matsusaka inequality holds. Hence we assume that $P_m(V) \leqq (n-1)(m^{n-1}+1)$ for all $2 \leqq m \leqq n+2$. Then there are at most a finite number of such polynomials in the form $P_m(V)$. Thus we can find a number l dependent on $\dim V$ only such that Matsusaka inequality holds for all $m>l$.

Case 2. $\gamma > n-1$. We denote m-genera $P_m(V)$ by $P(m)$ and assume $P(m) < \gamma(m^{n-1}+1)$ for all $m \in [2, n+2]$. If not, Matsusaka inequality holds for one value m such that $2 \leqq m \leqq n+2$. Thus we shall show that there exists a number l dependent only on n such that $P(m) > \gamma m^{n-1} + n-1$ for all $m \geqq l$, under the assumption. We construct Lagrangean interpolation function g of degree n with the same leading coefficient as $P(m)$ such that the polynomial equation $P(m)-g(m)=0$ has $n-2$ roots $<n$, and a root$>n$ and that $P(m)>g(m)$ for all $m \geqq n+2$. Moreover $h(m):=g(m)/\gamma$ is a polynomial in m whose coefficients depend only on n. Thus, $g(m)-\gamma(m^{n-1}+1)=0$ is equivalent to $h(m)-(m^{n-1}+1)=0$. Then we shall show that there exists a number l dependent only on n such that Matsusaka inequality holds for all $m \geqq l$. We put $g(i) = \gamma(i^{n-1}+1)$ when $i \equiv n+2 \bmod 2$ and $i \neq n+2$ for all i such that $2 \leqq i \leqq n+2$. Further, we put $g(i)=0$ if $i \equiv n+1 \bmod 2$, and $g(n+2)=\alpha\gamma$ for $2 \leqq i \leqq n+2$. Here, α is determined by the following equation

$$\textstyle\sum_{i=2}^{n+1} g(i)/(i-2)(i-3)\cdots(i \wedge i)\cdots(i-n-2)+\alpha\gamma/n! = \gamma/n!.$$

Note that α is a function of n. We claim that $P(m)>g(m)$ for all $m \geqq n+2$. Put $Q(m)=g(m)-P(m)$. Consider each interval $(i-1, i+1)$ for $i \equiv n+1 \bmod 2$, contained in $[2, n+1]$. Then $Q(i-1)>0$, $Q(i+1)>0$ and $Q(i)=-P(i) \leqq 0$ by definition. Hence $Q(m)=0$ has at least two roots or a double root in the open interval $(i-1, i+1)$. Moreover, it has at least one root in $(i-1, i]$ and also another in $[i, i+1)$.

Now, divide into two cases.

(a) $3 \equiv n+1 \bmod 2$. We have $(n-2)/2$ intervals in the form $(i-1, i+1)$; more precisely they are $(2, 4)$, $(4, 6)$, $\cdots$, $(n-2, n)$. Hence $Q(m)=0$ has $2(n-2)/2+1$ $(=n-1)$ roots in $(2, n+1]$.

(b) $2 \equiv n+1 \bmod 2$. We have $(n-3)/2$ intervals in the form $(i-1, i+1)$; these are $(3, 5)$, $(5, 7)$, $\cdots$, $(n-2, n)$. Thus $Q(m)=0$ has at least $1+2(n-3)/2+1$ $(=n-1)$ roots in $[2, n+1]$.

In each case, note that Q has degree $n-1$. Let $m_1, \cdots, m_{n-1}$ be all roots of $Q(m)=0$. Hence $Q(m)=a(m-m_1)\cdots(m-m_{n-1})$. Clearly $m_1, \cdots, m_{n-2}<n$ and $m_{n-1}>n$. From $Q(n)>0$, $a<0$. Thus $Q(m)<0$ for all $m>n+1$, i.e. $P(m)>g(m)$. Hence $P(m) > g(m) \geqq \gamma(m^{n-1}+1) > \gamma m^{n-1}+n-1$ for all integers $m \geqq \operatorname{Max}\{$maximal real root of $h(m)-(m^{n-1}+1)=0, n+2\}$. Thus, our proof is complete.

Remark. It is rather easy to verify that $10^{10} n^{10(n+2)}$ satisfies the condition of our Theorem.

References

[1] Matsusaka, T.: Polarized varieties with a given Hilbert polynomial. Amer. J. Math., **94**, 1027 (1972).
[2] Wilson, P.: The pluricanonical maps on varieties of general type. Bull. Lond. Math. Soc., **12**, 103–107 (1980).