54. *Singularities of Solutions of the Hyperbolic Cauchy Problem in Gevrey Classes*

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1. Introduction. Singularities of solutions of the hyperbolic Cauchy problem have been investigated by many authors. In these works, Hamilton flows (null bicharacteristic flows) played a key role. However, in general, Hamilton flows can not be defined unless the characteristic roots are smooth. In [8], we generalized Hamilton flows. In this note, we shall give outer estimates of the wave front sets in Gevrey classes of solutions of the Cauchy problems for hyperbolic operators, whose principal parts have real analytic coefficients, using the generalized Hamilton flows.

2. Assumptions and results. Let $P(x, \xi)$ be a polynomial of $\xi=(\xi_1, \xi')=(\xi_1, \cdots, \xi_n)$ and write $P(x, \xi)=\sum_{j=0}^m P_j(x, \xi)$, where $P_j(x, \xi)$ is a homogeneous polynomial of degree j in ξ. Let $1<\kappa<\infty$ and denote by $\mathcal{E}^{(\kappa)}$, $\mathcal{E}^{\{\kappa\}}$, $\mathcal{D}^{(\kappa)}$ and $\mathcal{D}^{\{\kappa\}}$ spaces of ultradifferentiable functions on $\boldsymbol{R}^n$ (Gevrey classes) (see, e.g., Komatsu [5]). Moreover we denote by $\mathcal{D}^{(\kappa)\prime}$ and $\mathcal{D}^{\{\kappa\}\prime}$ spaces of ultradistributions (see [5]). Now let us state our assumptions:

(A-1) The coefficients of $P_m(x, \xi)$ are real analytic, defined for $x=(x_1, \cdots, x_n) \in \boldsymbol{R}^n$ and the coefficients of $P_j(x, \xi)$ $(j=0, \cdots, m-1)$ belong to $\mathcal{E}^{\{\kappa_1\}}$, where $1<\kappa_1<\infty$.

(A-2) $P_m(x, \xi)$ is hyperbolic with respect to $\vartheta=(1, 0, \cdots, 0) \in \boldsymbol{R}^n$, i.e.,

$$P_m(x, \xi-i\tau\vartheta)\neq 0 \qquad \text{for } x \in \boldsymbol{R}^n,\ \xi \in \boldsymbol{R}^n \text{ and } \tau>0.$$

(A-3) $1<\kappa_1<\kappa_0\equiv r/(r-1)$, where $r\geqq 2$ and the multiplicities of the roots of $P_m(x, \xi_1, \xi')=0$ in ξ_1 are not more than r when $x \in \boldsymbol{R}^n$ and $\xi' \in \boldsymbol{R}^{n-1}\setminus\{0\}$.

We shall consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P(x, D)u(x)=f(x), & x \in \boldsymbol{R}^n, \\ \operatorname{supp} u \subset \{x_1 \geqq 0\}, \end{cases}$$

where $f \in \mathcal{D}^{\{\kappa_1\}\prime}$ with $\operatorname{supp} f \subset \{x_1 \geqq 0\}$ and $D=-i(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. First let us define the localization $P_{mz}(\delta z)$ at $z \in T^*\boldsymbol{R}^n$ by

$$P_m(z+s\delta z)=s^\mu(P_{mz}(\delta z)+o(1)) \qquad \text{as } s\rightarrow 0,$$

where $P_{mz}(\delta z)\not\equiv 0$ (in δz) is a (homogeneous) polynomial of $\delta z \in T_z(T^*\boldsymbol{R}^n)$. Then $P_{mz}(\delta z)$ is hyperbolic with respect to $(0, \vartheta) \in \boldsymbol{R}^{2n}$ (see [8]). Therefore we can define $\Gamma(P_{mz}, (0, \vartheta))$ as the connected component of the set

$\{\delta z \in T_z(T^*\boldsymbol{R}^n)\,;\, P_{mz}(\delta z) \neq 0\}$ which contains $(0, \vartheta)$. Define

$$\Gamma_z = \Gamma(P_{mz}, (0, \vartheta)) \subset T_z(T^*\boldsymbol{R}^n),$$
$$\Gamma_z^\sigma = \{(\delta x, \delta\xi) \in T_z(T^*\boldsymbol{R}^n)\,;\, \sigma((\delta x, \delta\xi), (\delta x^1, \delta\xi^1))$$
$$(= \delta x \cdot \delta\xi^1 - \delta x^1 \cdot \delta\xi) \geqq 0 \text{ for any } (\delta x^1, \delta\xi^1) \in \Gamma_z\}.$$

Now we can define "flow" $K_z^\pm$ emanating from z by

$$K_z^\pm = \{z(t) \in T^*\boldsymbol{R}^n\,;\, \{z(t)\} \text{ is a Lipschitz continuous curve satisfying}$$
$$(d/dt)z(t) \in \Gamma_{z(t)}^\sigma \text{ (a.e. } t) \text{ and } z(0) = z, \text{ and } \pm t \geqq 0\}$$

(see [8]). If P_m is strictly hyperbolic and $z \in T^*\boldsymbol{R}^n \backslash 0$ with $P_m(z) = 0$, then $K_z^\pm$ is the half bicharacteristic (the Hamilton flow) emanating from z along which $\pm x_1$ increases. Moreover, if $z \in T^*\boldsymbol{R}^n \backslash 0$, then $K_z^\pm \subset T^*\boldsymbol{R}^n \backslash 0$. We can also write

$$K_{(x,0)}^\pm = K_x^\pm \times \{0\},$$

where $K_x^\pm = \{x(t)\,;\, \{x(t)\}$ is a Lipschitz continuous curve satisfying $(d/dt)x(t) \in \Gamma(P_m(x(t), \cdot), \vartheta)^*$ (a.e. t) and $x(0) = x$, and $\pm t \geqq 0\}$ and $\Gamma^* = \{\delta x\,;\, \delta x \cdot \delta\xi \geqq 0$ for any $\delta\xi \in \Gamma\}$.

Definition. Let $1 < \kappa < \infty$ and $f \in \mathcal{D}^{(s)\prime}$, where $1 < s < \infty$. $WF_{(\kappa)}(f)$ (resp. $WF_{\{\kappa\}}(f)$) is defined as the complement in $T^*\boldsymbol{R}^n \backslash 0$ of the collection of all (x^0, ξ^0) in $T^*\boldsymbol{R}^n \backslash 0$ such that there are a neighborhood U of x^0 and a conic neighborhood Γ of ξ^0 such that for every $\phi \in \mathcal{D}^{(s)}$ with $\operatorname{supp} \phi \subset U$ and every $A > 0$ there is a positive number C (resp. for every $\phi \in \mathcal{D}^{(s)}$ with $\operatorname{supp} \phi \subset U$ there are positive numbers A and C) satisfying

$$|\mathcal{F}[\phi f](\xi)| \leqq C \exp[-A|\xi|^{1/\kappa}] \qquad \text{for } \xi \in \Gamma,$$

where $\mathcal{F}[f](\xi) \equiv \hat{f}(\xi)$ denotes the Fourier transform of f (see [3], [7]).

In addition to (A-1)–(A-3), we assume that

(A-4) $K_x^- \cap \{x_1 \geqq 0\}$ is compact for every $x \in \boldsymbol{R}^n$.

Theorem 1. *Assume that* (A-1)–(A-4) *are satisfied. If* $u \in \mathcal{D}^{\{\kappa_1\}\prime}$ *satisfies the Cauchy problem* (CP), *then*

$$WF_*(u) \subset \{z \in T^*\boldsymbol{R}^n \backslash 0\,;\, z \in K_{z^1}^+ \text{ for some } z^1 \in WF_*(f)\}$$
$$\equiv \mathcal{C} \circ WF_*(f),$$

where $*$ *denotes* (κ) *or* $\{\kappa\}$, *and* $\kappa_1 < \kappa < \kappa_0$ *if* $* = (\kappa)$, *and* $\kappa_1 \leqq \kappa < \kappa_0$ *if* $* = \{\kappa\}$.

Remark. Well-posedness of the Cauchy problem in $\mathcal{E}^*$ (or $\mathcal{D}^{*\prime}$) was proved by Ivrii [4], Bronshtein [2] and Trepreau [6], that is, the Cauchy problem (CP) has a unique solution $u \in \mathcal{D}^{*\prime}$ if $f \in \mathcal{D}^{*\prime}$.

Corollary. *Assume that* (A-1)–(A-4) *are satisfied and that the Cauchy problem* (CP) *is* C^∞ *well-posed. If* $f \in \mathcal{D}'$ *and* $u \in \mathcal{D}'$ *satisfies the Cauchy problem* (CP), *then*

$$WF(u) \subset \mathcal{C} \circ WF(f).$$

Theorem 2. *Assume that* (A-2)–(A-4) *are satisfied and that*

(A-1)′ *the coefficients of* $P(x, \xi)$ *belong to* $\mathcal{E}^{\{\kappa_1\}}$.

If $u \in \mathcal{D}^{\{\kappa_1\}\prime}$ *satisfies the Cauchy problem* (CP), *then*

$$\operatorname{supp} u \subset \{x \in \boldsymbol{R}^n\,;\, x \in K_y^+ \text{ for some } y \in \operatorname{supp} f\}.$$

3. Sketch of proof of Theorem 1. We shall give a sketch of the proof of Theorem 1, as Theorem 2 can be easily proved by the same argument as in the proof of Theorem 1.

Lemma 1. *Let $z \in T^*\boldsymbol{R}^n$. For every compact set M in Γ_z, there is a neighborhood U of z in $T^*\boldsymbol{R}^n$ such that $M \subset \Gamma_{z^1}$ for $z^1 \in U$.*

It follows from Lemma 1 that $K_z^{\pm}$ can be constructed approximately. In order to prove Theorem 1, it suffices to prove that for $z^0 \in T^*\boldsymbol{R}^n \backslash 0$ and every compact set M in Γ_{z^0} there is $z \in U \cap (\{z^0\} - M^\sigma)$ such that $z \in WF_*(u)$ and $z \neq z^0$, if $f \in \mathcal{D}^{\{\kappa_1\}\prime}$, $\operatorname{supp} f \subset \pi(U)$, $WF_*(f) \cap (\{z^0\} - M^\sigma) = \emptyset$, $P(x, D)u = f$, $\operatorname{supp} u \subset \{x_1 \geqq c\}$ and $z^0 \in WF_*(u)$, where U is a sufficiently small neighborhood of z^0, $\pi : T^*R^n \ni (x, \xi) \mapsto x \in \boldsymbol{R}^n$ and c is a real number. By successive iteration, we can construct a solution u of (CP) for $f \in \mathcal{D}^{\{\kappa_1\}\prime}$ with compact support, using estimates obtained by Bronshtein [2]. Then we can prove the above assertion by the same arguments as in [1], [7] and [8]. However, the proof is more complicate and we must use almost analytic extension and change their integral paths in (x, ξ)-space. In doing so, we need the following

Lemma 2. *Let $(x^0, \xi^0) \in T^*\boldsymbol{R}^n$ and M be a compact set in $\Gamma_{(x^0, \xi^0)}$. Then there are a neighborhood U of (x^0, ξ^0) in $T^*\boldsymbol{R}^n$ and $t_0 > 0$ such that*

$$P_m(x - ity, \xi - it\eta) \neq 0 \qquad \text{if } (x, \xi) \in U,\ (y, \eta) \in M \text{ and } 0 < t \leqq t_0.$$

Remark. Lemma 1 can be proved, using Lemma 2 and Rouché's theorem.

Lemma 3. *Let $(x^0, \xi^0) \in T^*\boldsymbol{R}^n \backslash 0$ and M be a compact set in $\Gamma_{(x^0, \xi^0)}$. Then there are a neighborhood U of (x^0, ξ^0), $C_M > 0$ and $t_0 > 0$ such that*

$$|P_{m(\beta)}^{(\alpha)}(x - ity, \xi - it\eta) / P_m(x - ity, \xi - it\eta)| \leqq C_M t^{-|\alpha| - |\beta|},$$

if $(x, \xi) \in U$, $(y, \eta) \in M$, $0 < t \leqq t_0$ and $|\alpha| + |\beta| \leqq m$, where $P_{m(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha \partial_x^\beta P_m(x, \xi)$.

In the forthcoming paper [9], we shall give a detailed proof. Finally we note that Theorems 1 and 2 are valid for hyperbolic systems.

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