7. *Zeros, Eigenvalues and Arithmetic*

By Akio FUJII
Department of Mathematics, Rikkyo University

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Let γ run over the imaginary parts of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Let $1/4+r^2$ run over the eigenvalues of the discrete spectrum of the Laplace-Beltrami operator in L^2 (the upper half plane $/\Gamma$), where we take $\Gamma=PSL(2, Z)$. Let α be a positive number. Here we introduce the zeta functions defined by

$$Z_\alpha(s)=\sum_{\gamma>0}\frac{\sin(\alpha\gamma)}{\gamma^s} \quad \text{and} \quad \mathfrak{Z}_\alpha(s)=\sum_{r>0}\frac{\sin(\alpha r)}{r^s}.$$

We are concerned with their analytic properties and their arithmetic.

To state our results we shall introduce some notations. $\Lambda(\cdot)$ is the von Mangoldt function. Let $\{P_0\}$ run over the primitive hyperbolic conjugacy classes in $PSL(2, Z)$. $N(P_0)$ denotes the square of the eigenvalue (>1) of a representative P_0. For a hyperbolic conjugacy class $\{P\}$ satisfying $P=P_0^k$ with a natural number k, we put $\tilde{\Lambda}(P)=(\log N(P_0))/(1-N(P)^{-1})$, where $N(P)=N(P_0)^k$. $A(\Gamma)$ denotes the area of the fundamental domain of Γ, which is equal to $\pi/3$. We assume the Riemann Hypothesis to get the results on γ or on $Z_\alpha(s)$. The following theorem describes a property of the distribution of γ or r.

Theorem 1. *Let $T>T_0$ and α be a positive number. Then*

$$\text{i)}\quad \sum_{0<\gamma\leq T} e^{i\alpha\gamma}=-\frac{1}{2\pi}\frac{\Lambda(e^\alpha)}{e^{\alpha/2}}T+\frac{e^{i\alpha T}}{2\pi i\alpha}\log T+O\left(\frac{\log T}{\log\log T}\right)$$

and

$$\text{ii)}\quad \sum_{0<r\leq T} e^{i\alpha r}=\frac{1}{\pi}\frac{\Lambda(e^{\alpha/2})}{e^{\alpha/2}}T+\frac{A(\Gamma)}{2\pi i\alpha}Te^{i\alpha T}+\frac{e^{-\alpha/2}}{2\pi}\Big(\sum_{N(P)=e^\alpha}\tilde{\Lambda}(P)\Big)T$$
$$+O\left(\frac{T}{\log T}\right).$$

We remark that i) is a refinement of Landau's theorem and has been proved by the author in [3]. ii) can be proved by the same method. Venkov [11] has studied the asymptotic behavior of the sum $\sum_{r>0}\cos(\alpha r)e^{-tr^2}$ as $t\to+0$. We see by this theorem that for any positive α as $m\to\infty$, $\sum_{0<\gamma\leq m}\sin(\alpha\gamma)/\gamma^s$ converges to $Z_\alpha(s)$ if $\mathrm{Re}\, s>0$ and $\sum_{0<r\leq m}\sin(\alpha r)/r^s$ converges to $\mathfrak{Z}_\alpha(s)$ if $\mathrm{Re}\, s>1$. Using the Poisson summation formula and the Selberg trace formula, we can show the following theorem.

Theorem 2. *For any positive α, $Z_\alpha(s)$ and $\mathfrak{Z}_\alpha(s)$ are entire.*

We remark that $\sum_{\gamma>0}\gamma^{-s}$ has been studied by Guinand [5] and Delsarte [2]. $\sum_{r>0}(1/4+r^2)^{-s}$ has been studied by Minakshisundaram and Pleijel [9]. In a similar manner we can prove our Theorem 2 and also study the zeta functions $\sum_{\gamma>0}\cos(\alpha\gamma)/\gamma^s$ and $\sum_{r>0}\cos(\alpha r)/r^s$.

As is usual in the theory of numbers, the values of the zeta functions at $s=1$ play important roles. The explicit formula for $Z_\alpha(1)$ is well known (cf. (2.7) of Guinand [5]) and the oscillation of $Z_\alpha(1)$ as $\alpha\to\infty$ is essentially that of $-e^{-\alpha/2}/2\,(\sum_{n\leqq e^\alpha}\Lambda(n)-e^\alpha)$. On the contrary, the evaluation of $\mathfrak{Z}_\alpha(1)$ is more complicated. As a direct by-product of the proof of Theorem 2 above, we can express $\mathfrak{Z}_\alpha(1)$ explicitly. In particular, we obtain the following corollary.

Corollary 1. *As* $\alpha\to\infty$,

$$\mathfrak{Z}_\alpha(1)=\frac{1}{2}e^{-\alpha/2}\Big(\sum_{N(P)\leqq e^\alpha}\tilde{\Lambda}(P)-e^\alpha\Big)+O(\alpha).$$

Thus we see that that $\mathfrak{Z}_\alpha(1)$ plays the same role in the theory of the distribution of $\tilde{\Lambda}(P)$ as does $Z_\alpha(1)$ in the prime number theory (cf. also Corollary 1′ below).

The values of the zeta functions at $s=0$ play also important roles. As by-products of the proof of Theorem 2, we can evaluate $Z_\alpha(0)$ or $\mathfrak{Z}_\alpha(0)$ explicitely. We mention only the following corollary.

Corollary 2. i) $\displaystyle\lim_{\alpha\to\log n}(\alpha-\log n)\,Z_\alpha(0)=-\frac{\Lambda(n)}{\sqrt{n}}$.

ii) $\displaystyle\lim_{\alpha\to 2\log n}(\alpha-2\log n)\mathfrak{Z}_\alpha(0)=\frac{1}{\pi}\frac{\Lambda(n)}{n}$ *and*

$$\lim_{\alpha\to\log N(P_1)}(\alpha-\log N(P_1))\mathfrak{Z}_\alpha(0)=\frac{1}{2\pi}\sum_{N(P)=N(P_1)}\frac{\tilde{\Lambda}(P)}{\sqrt{N(P)}},$$

where $\{P_1\}$ *is any hyperbolic conjugacy class.*

We remark here that if we use Kuznecov's version [7], [8] of Selberg's trace formula to prove Theorem 2, then we get another expression for $\mathfrak{Z}_\alpha(1)$ and $\mathfrak{Z}_\alpha(0)$. In particular, we obtain the following corollaries.

Corollary 1′. *As* $\alpha\to\infty$,

$$\mathfrak{Z}_\alpha(1)=\Big(\sum_{n\leqq e^{\alpha/2}}\cdot B(n)-e^{\alpha/2}\Big)+O(\alpha),$$

where $B(n)$ *is the residue at* $s=1$ *of the function* $\zeta(2s)\sum_{c=1}^\infty A_n(c)/c^s$ *and* $A_n(c)$ *is the number of the solutions of* $x^2+nx+1\equiv 0\pmod{c}$.

Corollary 2′. *For* $n\geqq 3$,

$$\lim_{\alpha\to 2\log(n+\sqrt{n^2-4})/2}\left(\alpha-2\log\frac{n+\sqrt{n^2-4}}{2}\right)\mathfrak{Z}_\alpha(0)=\frac{1}{\pi}B(n).$$

We can rewrite the second part of ii) of Corollary 2 and Corollary 2′ in the following form.

Corollary 3. *Let* n *be an integer* $\geqq 3$. *Suppose that* $n^2-4=Q^2D$

and D is square free. Let χ be the character of the quadratic number field $\boldsymbol{Q}(\sqrt{n^2-4})$ and $L(s,\chi)$ be the Dirichlet L-function. Then

$$L(1,\chi)=\pi F(n)^{-1}\lim_{\alpha\to 2\log(n+\sqrt{n^2-4})/2}\left(\alpha-2\log\frac{n+\sqrt{n^2-4}}{2}\right)\mathfrak{Z}_\alpha(0)$$
$$=\frac{\log(n+\sqrt{n^2-4})/2}{\sqrt{n^2-4}}\Phi(n)F(n)^{-1},$$

where we put

$$F(n)=\left(1-\frac{1}{2}\chi(2)\right)\prod_{\substack{p|Q\\p>2}}\left(1-\frac{\chi(p)}{p}\right)v_2(1,n)\left(1+\frac{1}{2}\right)^{-1}\prod_{\substack{p|n^2-4\\p>2}}\left(1+\frac{1}{p}\right)^{-1}v_p(1,n)$$
$$v_p(1,n)=1+\sum_{k=1}^{\infty}\frac{A_n(p^k)}{p^k}$$

for a prime number p,

$$\Phi(n)=\sum_{d>0,d^2|n^2-4}\frac{h(n^2-4,2d)}{\nu(n,d)},\qquad h(n^2-4,2d)$$

is the number of the classes of the quadratic forms $ax^2+2bxy+cy^2$ such that $b^2-ac=n^2-4$ and $(a,2b,c)=2d$,

$$\nu(n,d)=\mathrm{Max}\,\{\nu\,;\nu\,|\,k \text{ and } (\eta_n^k-\eta_n^{-k})/(\eta_n^{k/\nu}-\eta_n^{-k/\nu})\,|\,d\},$$

η_n is the fundamental unit of $\boldsymbol{Q}(\sqrt{n^2-4})$ and $\eta_n^k=(n+\sqrt{n^2-4})/2$.

Thus we get some new expressions for $L(1,\chi)$. In fact, from the second expression for $L(1,\chi)$ in the above corollary, we can derive Dirichlet's class number formula for the real quadratic number fields.

Finally, we shall describe some special asymptotic behaviors of the partial sums $\sum_{0<\gamma\leq m}\sin(\alpha\gamma)/\gamma^\sigma$ or $\sum_{0<r\leq m}\sin(\alpha r)/r^\sigma$ as $m\to\infty$. Namely, we show that they present Gibbs's phenomenon at certain points. Our results may be described as follows.

Theorem 3. *Let p be a prime number, k be an integer $\geqq 1$ and m run over the integers $\geqq 1$.*

i–1). *For any σ in $0<\sigma<1$,*

$$\lim_{m\to\infty}\left(\frac{\pi}{m}\right)^{1-\sigma}\sum_{0<\gamma\leqq m}\frac{\sin(\gamma(\log p^k\pm\pi/m))}{\gamma^\sigma}=\mp\frac{1}{2\pi}\frac{\log p}{p^{k/2}}\int_0^\pi\frac{\sin t}{t^\sigma}dt$$

and

$$\lim_{\alpha\to\log p^k\pm 0}|\alpha-\log p^k|^{1-\sigma}Z_\alpha(\sigma)=\mp\frac{1}{2\pi}\frac{\log p}{p^{k/2}}\int_0^\infty\frac{\sin t}{t^\sigma}dt.$$

i–2).

$$\lim_{m\to\infty}\sum_{0<\gamma\leqq m}\frac{\sin(\gamma(\log p^k\pm\pi/m))}{\gamma}-Z_{\log p^k}(1)=\mp\frac{1}{2\pi}\frac{\log p}{p^{k/2}}\int_0^\pi\frac{\sin t}{t}dt$$

and

$$\lim_{\alpha\to\log p^k\pm 0}Z_\alpha(1)-Z_{\log p^k}(1)=\mp\frac{1}{2\pi}\frac{\log p}{p^{k/2}}\int_0^\infty\frac{\sin t}{t}dt.$$

i–3). *For any σ in $0<\sigma\leqq 1$,*

$$\lim_{m\to\infty}\left(\frac{\pi}{m}\right)^{1-\sigma}\frac{1}{\log(m/\pi)}\sum_{0<\gamma\leqq m}\frac{\sin(\pm\gamma(\pi/m))}{\gamma^{\sigma}}=\pm\frac{1}{2\pi}\int_0^{\pi}\frac{\sin t}{t^{\sigma}}dt$$

and

$$\lim_{\alpha\to\pm 0}|\alpha|^{1-\sigma}\frac{1}{\log|1/\alpha|}Z_{\alpha}(\sigma)=\pm\frac{1}{2\pi}\int_0^{\infty}\frac{\sin t}{t^{\sigma}}dt.$$

ii). *For any* σ *in* $1<\sigma\leqq 2$,

$$\lim_{m\to\infty}\left(\frac{\pi}{m}\right)^{2-\sigma}\sum_{0<r\leqq m}\frac{\sin(\pm(\pi/m)r)}{r^{\sigma}}=\pm\frac{A(\Gamma)}{2\pi}\int_0^{\pi}\frac{\sin t}{t^{\sigma-1}}dt$$

and

$$\lim_{\alpha\to\pm 0}|\alpha|^{2-\sigma}\mathfrak{Z}_{\alpha}(\sigma)=\pm\frac{A(\Gamma)}{2\pi}\int_0^{\infty}\frac{\sin t}{t^{\sigma-1}}dt.$$

We remark that the details on the zeros of $\zeta(s)$ have appeared in [4] and the details on the eigenvalues of the Laplace-Beltrami operator will appear elsewhere. For comparison we have stated both results at the same time.

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