47. Vanishing Theorems in Asymptotic Analysis. II

By Hideyuki MAJIMA
Department of Mathematics, Hitotsubashi University

(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1984)

Let M be a complex manifold and let H be a divisor on M. For simplicity, suppose that the divisor H has only normal crossings. Denote by $\mathcal{O}$ the sheaf of germs of holomorphic functions, by $\mathcal{O}(*H)$ the sheaf of germs of meromorphic functions which are holomorphic in M–H and have poles on H and by $\mathcal{O}_{M\hat{\restriction}H}$ the formal completion of $\mathcal{O}$ along H: $\mathcal{O}_{M\hat{\restriction}H}=\text{Proj}\lim_{k\to\infty}\mathcal{O}/\mathcal{J}_H^k$, where $\mathcal{J}_H$ is the nullstellen ideal of H. Let M^- be the real blowing up of M along H and let $pr: M^-\to M$ be the natural projection. Denote by $\mathcal{A}^-$ the sheaf of germs of functions strongly asymptotically developable, and denote by $\mathcal{A}'^-$ and $\mathcal{A}_0^-$ the sheaf of germs of functions strongly asymptotically developable to $\mathcal{O}_{M\hat{\restriction}H}$ and to 0, respectively. Then, we have the short exact sequence of sheaves:

$$0\longrightarrow\mathcal{A}_0^-\xrightarrow{i}\mathcal{A}'^-\xrightarrow{FA}pr^*(\mathcal{O}_{M\hat{\restriction}H})\longrightarrow 0,$$

from which we obtain the long exact sequence of cohomologies:

$$\begin{aligned}0&\longrightarrow H^0(p^-,\mathcal{A}_0^-|_{p^-})\xrightarrow{i_{p,0}}H^0(p^-,\mathcal{A}'^-|_{p^-})\longrightarrow H^0(p^-,pr^*(\mathcal{O}_{M\hat{\restriction}H})|_{p^-})\\&\longrightarrow H^1(p^-,\mathcal{A}_0^-|_{p^-})\xrightarrow{i_{p,1}}H^1(p^-,\mathcal{A}'^-|_{p^-})\longrightarrow H^1(p^-,pr^*(\mathcal{O}_{M\hat{\restriction}H})|_{p^-})\\&\longrightarrow H^2(p^-,\mathcal{A}_0^-|_{p^-})\longrightarrow\cdots\longrightarrow H^n(p^-,\mathcal{A}_0^-|_{p^-})\longrightarrow\cdots,\end{aligned}$$

where $p^-=pr^{-1}(p)$, and

$$\begin{aligned}0&\longrightarrow H^0(M^-,\mathcal{A}_0^-)\xrightarrow{i_0}H^0(M^-,\mathcal{A}'^-)\longrightarrow H^0(M^-,pr^*(\mathcal{O}_{M\hat{\restriction}H}))\\&\longrightarrow H^1(M^-,\mathcal{A}_0^-)\xrightarrow{i_1}H^1(M^-,\mathcal{A}'^-)\longrightarrow H^1(M^-,pr^*(\mathcal{O}_{M\hat{\restriction}H}))\\&\longrightarrow H^2(M^-,\mathcal{A}_0^-)\longrightarrow\cdots\longrightarrow H^n(M^-,\mathcal{A}_0^-)\longrightarrow\cdots.\end{aligned}$$

In the previous article [2], we assert that $i_{p,1}$ is a zero mapping for any point p in H, and that i_1 is a zero mapping if $H^1(M,\mathcal{O})=0$. Here, we assert moreover the following:

Theorem 1. *For any point p in H, $H^q(p^-,\mathcal{A}_0^-|_{p^-})=0$, $q=2,\cdots,n$. If $H^q(M,\mathcal{O})=0$ and $H^q(M,\mathcal{O}_{M\hat{\restriction}H})=0$, $q=1,\cdots,n$, then $H^q(M^-,\mathcal{A}_0^-)=0$, $q=2,\cdots,n$.*

In order to prove Theorem 1, we use the following soft resolution of the sheaf $\mathcal{A}_0^-$:

$$\mathcal{A}_0^-\longrightarrow\mathcal{P}_{0,0}^-\xrightarrow{d''}\mathcal{P}_{0,1}^-\xrightarrow{d''}\cdots\xrightarrow{d''}\mathcal{P}_{0,n}^-\xrightarrow{d''}0,$$

where $\mathcal{P}_{0,q}^-$ denotes the sheaf on M^- of germs of differential forms of type $(0,q)$ with coefficients infinitely differentiable and infinitely flat on $pr^{-1}(H)$. Notice that the direct image $pr_*(\mathcal{P}_{0,q}^-)$ on M coincides with the sheaf $\mathcal{J}^{0,q}_{(M,H)}$ of germs of differential forms of type $(0, q)$ with coefficients infinitely differentiable and infinitely flat on H. It is well known that the sheaf $\mathcal{O}$ has the soft resolution

$$\mathcal{O}\longrightarrow\mathcal{E}^{0,0}\xrightarrow{d''}\mathcal{E}^{0,1}\xrightarrow{d''}\cdots\xrightarrow{d''}\mathcal{E}^{0,n}\xrightarrow{d''}0,$$

and that "Poincaré's Lemma holds" for $(\mathcal{E}^{0,\cdot}, d'')$, where $\mathcal{E}^{0,q}$ denotes the sheaf of germs of differential forms of type $(0, q)$ with coefficients infinitely differentiable on M. Moreover, according to [7] and [1], $\mathcal{O}_{M\hat{\uparrow}H}$ has the resolution of the form

$$\mathcal{O}_{M\hat{\uparrow}H}\longrightarrow\mathcal{E}^{0,0}_{H}\xrightarrow{d''}\mathcal{E}^{0,1}_{H}\xrightarrow{d''}\cdots\xrightarrow{d''}\mathcal{E}^{0,n}_{H}\xrightarrow{d''}0,$$

and "Poincaré's Lemma holds" for $(\mathcal{E}^{0,\cdot}_{H}, d'')$, where $\mathcal{E}^{0,q}_{H}$ denotes the sheaf of germs of differential forms of type $(0, q)$ with coefficients infinitely differentiable in the sense of Whitney on H (prolongated by 0 outside of H). On the other side, by the Whitney's extension theorem ([6]), we have the short exact sequences

$$0\longrightarrow\mathcal{J}^{0,q}_{(M,H)}\longrightarrow\mathcal{E}^{0,q}\longrightarrow\mathcal{E}^{0,q}_{H}\longrightarrow 0,\qquad q=0, 1, \cdots, n.$$

From these facts, by using the formal de Rham theorem and the computation of cohomologies of double complex, we can easily deduce the first assertion of Theorem 1. If $H^q(M, \mathcal{O})=0$ and $H^q(M, \mathcal{O}_{M\hat{\uparrow}H})$ for $q=1, \cdots, n$, we see that "Poincaré's Lemma holds" for $(\Gamma(M, \mathcal{E}^{0,\cdot}), d'')$ and $(\Gamma(M, \mathcal{E}^{0,\cdot}_{H}), d'')$, respectively. Therefore, we can deduce the second assertion of Theorem 1.

Finally, we give some applications of Theorem 1.

Let $\mathcal{S}$ be a locally free sheaf of $\mathcal{O}(*H)$-modules of rank m and let ∇ be an integrable connection on $\mathcal{S}$. We view ∇ as a homomorphism of abelian sheaves

$$\nabla : \mathcal{S}\longrightarrow\mathcal{S}\otimes_{\mathcal{O}(*H)}\Omega^1,$$

which satisfies the Leibniz's rule and which extends to define a structure of complex on $\mathcal{S}\otimes_{\mathcal{O}(*H)}\Omega^\cdot$, the de Rham complex of $(\mathcal{S}, \nabla)$:

$$\mathcal{S}\xrightarrow{\nabla}\mathcal{S}\otimes_{\mathcal{O}(*H)}\Omega^1\xrightarrow{\nabla}\cdots\xrightarrow{\nabla}\mathcal{S}\otimes_{\mathcal{O}(*H)}\Omega^n\xrightarrow{\nabla}0,$$

where Ω^q denotes the sheaf of germs of holomorphic q-forms on M. For simplicity, we write $\mathcal{S}\Omega^\cdot$ for $\mathcal{S}\otimes_{\mathcal{O}(*H)}\Omega^\cdot$ and denote the de Rham complex by $(\mathcal{S}\Omega^\cdot, \nabla)$. Moreover, we write $\mathcal{S}_{f/c}\Omega^\cdot$ and $\mathcal{S}_0\Omega^\cdot$ for $\mathcal{S}\Omega^\cdot\otimes_{\mathcal{O}}\mathcal{O}_{M\hat{\uparrow}H}/\mathcal{S}\Omega^\cdot\otimes_{\mathcal{O}}\mathcal{O}_{M\hat{\uparrow}H}$ and $\mathcal{S}\Omega^\cdot\otimes_{pr^*\mathcal{O}}\mathcal{A}_0^-$, respectively, and denote the induced complexes by $(\mathcal{S}_{f/c}\Omega^\cdot, \nabla)$ any $(\mathcal{S}_0\Omega^\cdot, \nabla)$, respectively. We suppose that (H.1) or (H.2) is satisfied for and point on H, where (H.1) and (H.2) are the same as in the previous article [3]. Then, we have the following isomorphism theorems:

Theorem 2.

(1) *For any point p on H and for any $q=0, 1, \cdots, n$,*

(a) $\mathcal{H}^q(\mathcal{S}_{f/c}\Omega^\cdot, \nabla)_p\cong H^{q+1}(p^-, \mathcal{H}^0(\mathcal{S}^0\Omega^\cdot, \nabla)|_{p^-})$,

(b) $\mathcal{H}^q(\mathcal{S}\Omega^\cdot, \nabla)_p\cong H^q(p^-, \mathcal{H}^0(\mathcal{S}_0\Omega^\cdot, \nabla)|_{p^-})$,

(2) *If* $H^q(M, \mathcal{O})=0$ *and* $H^q(M, \mathcal{O}_{M\hat{|}H})=0$ *for* $q=1, \cdots, n$, *then for* $q=0, 1, \cdots, n$,

(a) $H^q(\Gamma(M, \mathcal{S}_{f/c}\Omega^{\cdot}), \nabla) \cong H^{q+1}(M^{-}, \mathcal{H}^0(\mathcal{S}_0\Omega^{\cdot}, \nabla))$,

(b) $H^q(\Gamma(M, \mathcal{S}\Omega^{\cdot}), \nabla) \cong H^q(M^{+}, \mathcal{H}^0(\mathcal{S}_0\Omega^{\cdot}, \nabla))$.

In [3], we stated only (2)-(b) for $q=1$.

Remark. In one-variable case, (1)-(a) in Theorem 2 is always valid. In the same case, if $\mathcal{H}^0(\mathcal{S}\Omega^{\cdot}\otimes_{\mathcal{O}}\mathcal{O}_{M\hat{|}H}, \nabla)_p=0$ for any point p on H, (1)-(b) and (2)-(b) in Theorem 2 are valid.

The detail will be published elsewhere.

Acknowledgements. The author would like to express his gratitude to Professor J.-P. Ramis for the suggestion that our notion of strongly asymptotically developability is identified with that of infinite differentiability in the sense of Whitney on the boundaries for functions holomorphic in polysectors. He also would like to express his gratitude to Professor B. Malgrange for the critics, which made some points correct.

References

[1] Bingener, J.: Über Formale Komplexe Räume. Manuscripta Math., 253–293 (1978) (cf. Math. Review 58 #11492 by C. Banica).

[2] Majima, H.: Vanishing theorems in asymptotic analysis. Proc. Japan Acad., **59**A, 146–149 (1983).

[3] ——: ∇-Poincaré's lemma and ∇-de Rham cohomology for an integrable connection with irregular singular points. ibid., 150–153 (1983).

[4] ——: Asymptotic analysis for integrable connections with irregular singular points. (1983) (preprint).

[5] Majima, H. and Sibuya, Y.: Cohomological characterization of regular singularities in several variables. (1984) (preprint).

[6] Malgrange, B.: Ideals of Differentiable Function. Oxford Univ. Press (1966).

[7] Ramis, J.-P.: Variation sur le Thème "GAGA". Lect. Notes in Math., vol. 694, Springer-Verlag, pp. 228–289 (1978).