53. *Representations of Weyl Group and Its Subgroups on the Virtual Character Modules*

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(Communicated by Kôsaku YOSIDA, M. J. A., June 12, 1984)

Let G be a connected reductive Lie group and $\mathfrak{g}$ its Lie algebra. We call G acceptable if there exists a connected complex Lie group G_C with Lie algebra $\mathfrak{g}_C=\mathfrak{g}\otimes_R C$ which has the following two properties. (1) The canonical injection from $\mathfrak{g}$ into $\mathfrak{g}_C$ can be lifted up to a homomorphism of G into G_C. (2) For a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$, let ρ be half the sum of positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Then $\xi_\rho(\exp X)=\exp(\rho(X))$ $(X\in\mathfrak{h}_C)$ defines a unique character of H_C into C^*.

Assume G is acceptable. Fix an infinitesimal character λ, and let $V(\lambda)$ be the virtual character module of G which has the infinitesimal character λ. If λ is regular, $V(\lambda)$ is equal to the module of all the invariant eigendistributions (IEDs) on G with eigenvalue λ. We always assume that λ is regular in the following.

Let H be a Cartan subgroup of G and $\mathfrak{h}$ its Lie algebra. We denote the Weyl group of $(\mathfrak{g}_C, \mathfrak{h}_C)$ by $W=W(\mathfrak{h}_C)$. In this paper, we define a natural representation of a subgroup $W_H(\lambda)$ of W on a subspace $V_H(\lambda)$ of $V(\lambda)$. And we clarify this $W_H(\lambda)$-module structure of $V_H(\lambda)$. Let Car (G) be the set of conjugacy classes of Cartan subgroups of G, and $[H]\in\mathrm{Car}(G)$ the conjugacy class of H. Then we have $V(\lambda)=\sum^{\oplus}_{[H]\in\mathrm{Car}(G)}V_H(\lambda)$ (see Theorem 1) and if λ is integral for G_C, $W_H(\lambda)$ is equal to W (Theorem 4). So, for integral λ, we can consider W-module structure of $V(\lambda)$. Representations of W on $V(\lambda)$ are considered in [1], [3] and so on. Our present definition of the representation of $W_H(\lambda)$ coincides essentially with their definitions in the case that λ is integral (Theorem 2). These representations of "integral Weyl groups" $W_H(\lambda)$'s may be useful to classify the irreducible admissible representations of G.

§1. Definition of representations of Weyl groups. At first we quote the results of T. Hirai [2]. Let H be a Cartan subgroup of G with Lie algebra $\mathfrak{h}$. Let $S(\mathfrak{h}_C)$ be the space of all the polynomial functions on $\mathfrak{h}_C=\mathfrak{h}\otimes_R C$ and $I(\mathfrak{h}_C)$ the space of all the elements in $S(\mathfrak{h}_C)$ which are invariant under the action of the Weyl group $W(\mathfrak{h}_C)$. Put $W_G(H)=N_G(H)/Z_G(H)$, where $N_G(H)$ denotes the normalizer of H in G and $Z_G(H)$ the centralizer. We denote by $\mathfrak{B}(H;\lambda)$ the set of analytic functions ζ on H satisfying the conditions (1) and (2).

(1) ζ is an eigenfunction of $I(\mathfrak{h}_C)$ with eigenvalue λ, where we identify elements of $I(\mathfrak{h}_C)$ with differential operators of constant coefficients on H.

(2) ζ is ε-symmetric under $W_G(H)$, i.e.,

$$\zeta(wh)=\varepsilon(w, h)\zeta(h) \qquad (h \in H, w \in W_G(H)),$$

where $\varepsilon(w, h)$ is defined as follows. An element $w \in W_G(H)$ naturally induces an element $\tilde{w}$ of $W(\mathfrak{h}_C)$. Let $N_I(\tilde{w})$ be the number of imaginary roots $\alpha>0$ for which $\tilde{w}^{-1}\alpha<0$, and $S_R(\tilde{w})$ the set of real roots $\alpha>0$ for which $\tilde{w}^{-1}\alpha<0$. Then we put for $h \in H$ and $w \in W_G(H)$,

$$\varepsilon(w, h)=(-1)^{N_I(\tilde{w})} \prod_{\alpha \in S_R(\tilde{w})} \operatorname{sgn}(\xi_{\tilde{w}^{-1}\alpha}(h)).$$

Here ξ_α is the character of H defined by the equation $\mathrm{Ad}(h)X_\alpha=\xi_\alpha(h)X_\alpha$ $(h \in H)$, where X_α is a non-zero root vector for α.

We can define natural order on Car (G) such that a maximal split Cartan subgroup is the smallest element with respect to this order ([2], p. 274). We say an IED θ has a height $[H]$ if $\theta|_H\not\equiv 0$ and $\theta|_{H'}\equiv 0$ for any H' such that $[H']>[H]$. We call θ extremal if θ has the unique height.

Theorem 1 ([2], p. 284, p. 307). (1) *For any element ζ of $\mathfrak{B}(H;\lambda)$, we can construct an extremal IED $T\zeta$ in $V(\lambda)$ which has the height $[H]$ and on H it naturally provides ζ* (see [2], p. 272).

(2) *Conversely, any element of $V(\lambda)$ can be written as a linear combination of IEDs which are of the form $T\zeta$ $(\zeta \in \mathfrak{B}(H;\lambda))$ for some H's.*

For the infinitesimal character $\lambda \in \mathfrak{h}_C{}^*$ and H, let $\tilde{W}_H(\lambda)$ be the set of $w \in W(\mathfrak{h}_C)$ for which $\exp\,(w\lambda, X)$ $(X \in \mathfrak{h})$ defines an analytic function on H_0, the identity component of H. Let L be the kernel of the map $\exp: \mathfrak{h}\to H_0$. Then $w \in W(\mathfrak{h}_C)$ belongs to $\tilde{W}_H(\lambda)$ if and only if $\langle w\lambda, L\rangle \subset 2\pi\sqrt{-1}\,\boldsymbol{Z}$, where $\langle,\rangle$ is the pairing of $\mathfrak{h}_C{}^*\times\mathfrak{h}_C$. Put

$$L_\lambda=\sum_{w \in \tilde{W}_H(\lambda)} w^{-1}L, \qquad W_H(\lambda)=\{w \in W(\mathfrak{h}_C)\,|\,wL_\lambda=L_\lambda\}.$$

Let $W(H_i)=\{\tilde{w}\,|\,w \in W_G(H_i)\}$, where $\{H_i\,|\,0\leqq i\leqq l\}$ is a set of representatives of conjugacy classes of connected components of H under $W_G(H)$. Then we get the following proposition.

Proposition 1. *The set $\tilde{W}_H(\lambda)$ is invariant under the left multiplication of $W(H_i)$ and the right multiplication of $W_H(\lambda)$. Moreover, the group $W_H(\lambda)$ is the largest subgroup of $W(\mathfrak{h}_C)$ which leaves $\tilde{W}_H(\lambda)$ invariant under the right multiplication.*

Proposition 2 ([2], p, 319). *The space $\mathfrak{B}(H;\lambda)$ has a basis consisting of the elements of the form: for $0\leqq i\leqq l$ and $\{t\}\subset\tilde{W}_H(\lambda)$ a complete system of representatives for $W(H_i)\backslash\tilde{W}_H(\lambda)$,*

$$\zeta_{i,t}(wa_i \exp X)=\varepsilon(w, a_i) \sum_{s \in W(H_i)} \varepsilon(s, a_i) \exp\,(t\lambda, sX) \quad (w \in W_G(H), X \in \mathfrak{h}),$$

and $\zeta_{i,t}$ is zero outside the $W_G(H)$-orbit of H_i, where a_i is a minimal *element in* H_i (see [2], p. 314).

Definition. For $u \in W_H(\lambda)$ and $\zeta_{i,t}$ above, put

$$(R_u\zeta_{i,t})(wa_i \exp X) = \varepsilon(w, a_i) \sum_{s \in W(H_i)} \varepsilon(s, a_i) \exp(tu^{-1}\lambda, sX).$$

Then we define a representation τ of $W_H(\lambda)$ on $V_H(\lambda) = \boldsymbol{T}(\mathfrak{B}(H;\lambda))$ by

$$\tau_u(\boldsymbol{T}\zeta_{i,t}) = \boldsymbol{T}(R_u\zeta_{i,t}).$$

In the case that G satisfies Zuckerman's conditions ([3], p. 498) and λ is integral, we get the following theorem.

Theorem 2. *Let G be a connected semisimple Lie group with finite centre. Suppose that G is acceptable and G_C simply connected. Then, if λ is integral, $W_H(\lambda) \cong W$ for any Cartan subgroup H of G and we get the representation τ of W on $V(\lambda) = \sum^{\oplus}_{[H] \in \mathrm{Car}(G)} V_H(\lambda)$. Moreover, τ is equivalent to the representation which Zuckerman defined* ([3], p. 499) *and there exists an intertwining operator between these two representations which is diagonal.*

This theorem shows that Hirai's method $\boldsymbol{T}$, which gives the map from $\sum^{\oplus}_{[H] \in \mathrm{Car}(G)} \mathfrak{B}(H;\lambda)$ onto $V(\lambda)$, commutes essentially with actions of Weyl group, the natural one in the former and Zuckerman's one in the latter.

By Theorem 2, we can relate τ to the functor $(\cdot) \otimes F$, where F is a finite dimensional representation of G, if G and λ satisfy the assumptions of Theorem 2.

§2. The $W_H(\lambda)$-module structure of $V_H(\lambda)$. We can prove the following main theorem.

Theorem 3. *Let $\Gamma \subset \tilde{W}_H(\lambda)$ be a complete system of representatives of a coset space $W(H_i) \backslash \tilde{W}_H(\lambda) / W_H(\lambda)$. Then as a $W_H(\lambda)$-module,*

$$V_H(\lambda) \cong \sum_{i=0}^{l}{}^{\oplus} \sum_{\gamma \in \Gamma}{}^{\oplus} \mathrm{Ind}_{W^{\gamma}}^{W_H(\lambda)} \varepsilon^{\gamma},$$

where $W^{\gamma} = W_H(\lambda) \cap \gamma^{-1} W(H_i)\gamma$ and ε^{γ} is a character of W^{γ} defined by $\varepsilon^{\gamma}(w) = \varepsilon(\gamma w \gamma^{-1}, a_i)$ $(a_i \in H_i, w \in W^{\gamma})$.

We say λ is integral for G_C if λ is the differential of a character of H_C. If λ is integral for G_C, then $\tilde{W}_H(\lambda) = W_H(\lambda) \cong W$ and this permits us to consider W-module structure of $V(\lambda) = \sum^{\oplus}_{[H] \in \mathrm{Car}(G)} V_H(\lambda)$. Using Theorem 3, we get the following theorem.

Theorem 4. *If λ is integral for G_C, W-module $V(\lambda)$ is decomposed as follows.*

$$V(\lambda) \cong \sum_{[H] \in \mathrm{Car}(G)}{}^{\oplus} \sum_{i=0}^{l}{}^{\oplus} \mathrm{Ind}_{W(H_i)}^{W} \varepsilon_i,$$

where $\{H_i \mid 0 \leqq i \leqq l\}$ is a set of representatives of conjugacy classes of connected components of H under $W_G(H)$, and ε_i is a character of $W(H_i)$ with values in $\{\pm 1\}$ defined by $\varepsilon_i(w) = \varepsilon(w, a_i)$ $(w \in W(H_i), a_i \in H_i)$.

This theorem is a generalization of Proposition 2.4 in [1].

The author expresses his hearty thanks to Prof. T. Hirai for many useful discussions and kind encouragements.

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