51. *Factorization of Entire Solutions of Some Difference Equations*

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1. **Introduction.** A meromorphic function $h(x)$ is said to be *factored* if there are a meromorphic function $f(x)$ and an entire function $g(x)$ such that $h(x)=f(g(x))$. $h(x)$ is said to be *pseudo-prime* if every such factorization $h=f\circ g$ implies that either f is rational or g is a polynomial [4]. In this paper, we will consider factorization of solutions of the equation

(1.1) $$y(x+1)=P(y(x)),$$

where $P(w)$ is a polynomial of degree $p\geqq 2$:

(1.2) $$P(w)=a_pw^p+\cdots+a_1w+a_0,\quad a_p\neq 0,\quad p\geqq 2.$$

Any meromorphic solution $y(x)$ of (1.1) is transcendental and entire, unless it is a constant [6], [7]. If there is a number λ such that

(1.3) $$\lambda=P(\lambda)\ \text{ and }\ P'(\lambda)=1,\quad P(w)=\lambda+(w-\lambda)+A_m(w-\lambda)^{m+1}+\cdots,$$

then the difference equation (1.1) possesses an entire solution $\phi_\lambda(x)$ which is expanded asymptotically

(1.3′) $$\phi_\lambda(x)\sim\lambda+x^{-1/m}\sum_{j+k\geqq 0}c_{jk}x^{-j/m}\left(\frac{\log x}{x}\right)^k$$

as x tends to ∞ through $D(R,\varepsilon)$:

(1.3″) $$D(R,\varepsilon)=\{|x|>R,\ |\arg x-\pi|<(\pi/2)-\varepsilon\}\cup\{\mathrm{Im}\,[xe^{-i\varepsilon}]>R\}\cup\{\mathrm{Im}\,[xe^{i\varepsilon}]<-R\}$$

where $\varepsilon>0$ is arbitrarily fixed, and $R(>0)$ depends on ε and c_{m0}, in which m is the integer in (1.3) ($A_m\neq 0$) [5], [7].

If there is a number λ such that

(1.4) $$\lambda=P(\lambda)\ \text{ and }\ |P'(\lambda)|>1,$$

then (1.1) possesses an entire solution $s_\lambda(x)$ which is expanded as

(1.4′) $$s_\lambda(x)=\lambda+\sum_{j=1}^\infty p_jb^{jx}=\psi_\lambda(b^x)\qquad(\text{we write } P'(\lambda) \text{ as b})$$

where

(1.4″) $$\psi_\lambda(t)=\lambda+\sum_{j=1}^\infty p_jt^j$$

is an entire solution of the Schröder equation

(1.1′) $$w(bt)=P(w(t)).$$

Further we have shown that any entire solution $y(x)$ of (1.1) satisfies

(1.5) $$y(x-\mu)\to\lambda\qquad\text{as}\quad\mu\uparrow\infty,$$

uniformly on any compact set, where λ is a number for which either (1.3) or (1.4) holds [7], [8]. If $y(x)$ satisfies (1.5) for a λ with (1.3), then $y(x)$ can be written as

(1.6) $$y(x)=\phi_\lambda(x+\kappa(x)),$$
where $\kappa(x)$ is an entire function with period 1, i.e., if we write $g(x) = x+\kappa(x)$, then $y(x)$ is factored as $y=\phi_\lambda \circ g$ and g satisfies

(1.6′) $$g(x+1)=g(x)+1.$$
If solution $y(x)$ satisfies (1.5) for a λ with (1.4), then

(1.7) $$y(x)=\psi_\lambda(\kappa(x)b^x),$$
where $\kappa(x)$ is also an entire function with period 1, i.e., if we write $g(x)=\kappa(x)b^x$, then we have $y=\psi_\lambda \circ g$ and g satisfies

(1.7′) $$g(x+1)=bg(x).$$

Thus, any solution of (1.1) can be factored with either ϕ_λ or ψ_λ and a function $g(x)$ which satisfies a difference equation (1.6′) or (1.7′). Thus it would be natural to ask whether $\phi_\lambda(x)$ would be factored as $\phi_\lambda=f\circ g$ in which $g(x)$ satisfies some difference equation

(1.8) $$g(x+1)=G(g(x)),$$
where $G(w)$ is an entire function. Similarly, we ask whether ψ_λ would be factored as $\psi_\lambda=f\circ h$ in which $h(t)$ satisfies some Schröder equation

(1.8′) $$h(bt)=G(h(t)), \qquad \text{with an entire } G(w).$$
In this direction, we prove here

Theorem 1. *$\phi_\lambda(x)$ can not be factored with transcendental f and g, where $g(x)$ satisfies* (1.8). *Similarly $\psi_\lambda(t)$ can not be factored with transcendental f and h, where $h(t)$ satisfies* (1.8′).

Theorem 2. *Let $y(x)$ be an entire solution of* (1.1). *Suppose $y(x)$ is factored as $y=f\circ g$ with transcendental f and g, where g satisfies some difference equation of the form* (1.8), *then we have*

(1.9) $$g(x)=ax+\kappa(x) \qquad (a\neq 0) \quad \text{or}$$

(1.9′) $$g(x)=\kappa(x)c^x+d \qquad (c\neq 0)$$
where $\kappa(x)$ is an entire function with period 1, *and a, c, d are constants.*

For the proof, we need the following lemma, which we proved in [9]. Professor W. K. Hayman kindly informed me that the lemma had been obtained by Dr. R. Goldstein [3]. His method is different from that in [9].

Lemma 1. *Let $P(w)$ and $Q(w)$ be polynomials of degree p and q, respectively, with $q\geqq 2$. Suppose there is a meromorphic function $f(x)$ with*

(1.10) $$f(Q(x))=P(f(x)).$$
Then we must have that $p=q$, and $f(x)$ can not be transcendental.

2. Proof of Theorem 2. Let $y(x)$ be a solution of (1.1) and let $y(x)=f(g(x))$ with transcendental f and g, with $g(x+1)=G(g(x))$, where $G(w)$ is an entire function. Suppose $G(w)$ be transcendental. Then, by a Theorem of Clunie [1, p. 77, Theorem 2 (ii)], we obtain

(2.1) $$\limsup_{r\to\infty} [T(r, f\circ G)/T(r, f)] = \infty.$$

On the other hand, we get

$$T(r, f\circ G) = T(r, P\circ f) = pT(r, f) + 0\ (1)$$

by [2, p. 47, Theorem 6.1], which contradicts with (2.1). Thus $G(w)$ can not be transcendental and must be a polynomial.

By Lemma 1, $G(w)$ can not be of degree $\geqq 2$, since $f(x)$ is transcendental by assumption. Therefore

$$G(w) = cw + a.$$

Suppose $c=1$. If $a=0$, then $g(x)$ and hence $y(x)=f(g(x))$ would be periodic, which is a contradiction. Hence, if $c=1$, then $a\neq 0$. Thus we have that $g(x+1)=g(x)+a$, and hence $g(x)=ax+\kappa(x)$, with an entire periodic function $\kappa(x)$ and a constant $a\neq 0$.

Suppose $c\neq 1$. Then $g(x+1)=cg(x)+a$, hence $g(x)=\kappa(x)c^x+d$ with $d=a/(1-c)$. Thus we obtain the assertion of the Theorem.

3. Proof of Theorem 1. We note that $\phi_\lambda(x)$ is univalent in a half-plane

$$H=\{x\,;\,\mathrm{Re}\,x< -R_1\}$$

for sufficiently large R_1 [7]. Suppose $\phi_\lambda(x)=f(g(x))$ with transcendental f and g, where g satisfies an equation of the form (1.8). By Theorem 2, $g(x)$ must be written as either in (1.9) or in (1.9′). In either case, $f(g(x))$ can not be univalent in H unless $\kappa(x)\equiv$const. In fact, if $\kappa(x)\not\equiv$const., then there are x_1 and x_2 such that $x_1\neq x_2$ and $ax_1+\kappa(x_1)=ax_2+\kappa(x_2)$. Take a natural number k so large that $x'_j=x_j-k\in H$, $j=1,2$. Then we would have $f(g(x'_1))=f(g(x'_2))$, $x'_1\neq x'_2$, and $f(g(x))$ is not univalent in H. The possibility that $g(x)=a_1c^x+d$ (a_1=a const.) is also excluded by the univalency of $f(g(x))$ in H.

$\psi_\lambda(t)$ is univalent in $|t|<\rho_1$ for a small ρ_1. Suppose $\psi_\lambda(t)=f(h(t))$ with transcendental f and h, where $h(t)$ satisfies an equation of the form (1.8′), with $b=P'(\lambda)$. Put $g(x)=h(b^x)$. Then $g(x)$ satisfies a difference equation (1.8). By Theorem 2, we have

either (i) $\quad g(x)=h(b^x)=ax+\kappa(x)$

or (ii) $\quad g(x)=h(b^x)=\kappa(x)c^x+d.$

Suppose (i). Then

$$h(t)=\frac{a}{\log b}\log t+\kappa\left(\frac{\log t}{\log b}\right).$$

Since $h(t)$ must be entire, the case (i) can not occur.

Suppose (ii). Then

$$h(t)=\kappa\left(\frac{\log t}{\log b}\right)t^\alpha+d, \qquad \alpha=\frac{\log c}{\log b}.$$

Since $h(t)$ must be holomorphic and univalent in $|t|<\rho_1$, we have that $\alpha=1$, $c=b$. Further, $\kappa(x)$ must be a constant by the univalency of $\psi_\lambda(t)$. Thus we obtain the result.

4. Remark. So far we have been concerned with factorization $f \circ g$ assuming that $g(x)$ satisfies an equation (1.8). One may conjecture that, if a solution $y(x)$ of (1.1) would be factored as $y = f \circ g$, then g would satisfy an equation of the form (1.8). This is not true if f is admitted to be rational. For example, consider the equation

$$y(x+1) = 4y(x)(1-y(x)). \tag{4.1}$$

A solution $y(x) = \sin^2(2^x)$ is factored as $f \circ g$ with $f(z) = z^2$ and $g(x) = \sin(2^x)$, and $g(x)$ satisfies

$$g(x+1) = G(g(x))$$

with $G(w) = 2w(1-w^2)^{1/2}$, which is not one-valued. If we require for f to be transcendental, then we do not know whether the conjecture would be true or not.

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