91. *On the Algebra of Absolutely Convergent Disk Polynomial Series*

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Let $\alpha \geqq 0$ and let m, n be nonnegative integers. *Disk polynomials* $R_{m,n}^{(\alpha)}$ are defined in terms of Jacobi polynomials by

$$R_{m,n}^{(\alpha)}(z)=\begin{cases} R_n^{(\alpha, m-n)}(2r^2-1)\, e^{i(m-n)\theta} r^{m-n} & \text{if } m \geqq n, \\ R_m^{(\alpha, n-m)}(2r^2-1)\, e^{i(m-n)\theta} r^{n-m} & \text{if } m<n, \end{cases}$$

where $z=re^{i\theta}$ and $R_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n and of order (α, β) normalized so that $R_n^{(\alpha,\beta)}(1)=1$. If $\alpha=q-2$, $q=2,3,4,\cdots$, then disk polynomials are the spherical functions on the sphere S^{2q-1} considered as the homogeneous space $U(q)/U(q-1)$. Let D and $\bar{D}$ be the open unit disk and the closed unit disk in the complex plane, respectively. Denote by $A^{(\alpha)}$ the space of absolutely convergent disk polynomial series on $\bar{D}$, that is, the space of functions f on $\bar{D}$ such that

$$f(z)=\sum_{m,n=0}^{\infty} a_{m,n} R_{m,n}^{(\alpha)}(z) \quad \text{with} \quad \sum |a_{m,n}|<\infty,$$

and introduce a norm to $A^{(\alpha)}$ by $\|f\|=\sum |a_{m,n}|$.

The purpose of this note is to study the structure of the space $A^{(\alpha)}$. Details will be published elsewhere.

1. Firstly we mention some properties of $R_{m,n}^{(\alpha)}$:

(i) $R_{m,n}^{(\alpha)}(z)$ is a polynomial of degree $m+n$ in x and y where $z=x+iy$.

(ii) $\displaystyle\int_{\bar{D}} R_{m,n}^{(\alpha)}(z) R_{k,l}^{(\alpha)}(\bar{z})\, dm_\alpha(z)=h_{m,n}^{(\alpha)-1} \delta_{mk} \delta_{nl}$,

where $dm_\alpha(z)=\left(\frac{\alpha+1}{\pi}\right)(1-x^2-y^2)^\alpha\, dx\, dy$, $h_{m,n}^{(\alpha)}=(m+n+\alpha+1)\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)\{(\alpha+1)\Gamma(\alpha+1)^2\Gamma(m+1)\Gamma(n+1)\}^{-1}$, $\bar{z}=x-iy$ and δ_{mk} is Kronecker's δ.

(iii) $|R_{m,n}^{(\alpha)}(z)| \leqq 1$ on $\bar{D}$ ([7; (5.1)]).

(iv) $R_{m,n}^{(\alpha)}(z) R_{k,l}^{(\alpha)}(z)=\sum_{p,q} c_{p,q}(m, n; k, l) h_{p,q}^{(\alpha)} R_{p,q}^{(\alpha)}(z)$ with $c_{p,q}(m, n; k, l) \geqq 0$ ([8; Corollary 5.2]).

Disk polynomials are studied by several authors and we cite here only T. H. Koornwinder [7].

The space $A^{(\alpha)}$ consists of continuous functions on $\bar{D}$ since if $\sum |a_{m,n}|<\infty$ then the series $\sum a_{m,n} R_{m,n}^{(\alpha)}(z)$ converges uniformly on $\bar{D}$ by (iii). Let l^1 be the Banach space of absolutely convergent double sequences $b=\{b_{m,n}\}_{m,n=0}^{\infty}$ with norm $\|b\|=\sum |b_{m,n}|$. Then $A^{(\alpha)}$ is a

Banach space isometric to l^1 by the map $f\to\{\hat{f}(m,n)h_{m,n}^{(\alpha)}\}_{m,n=0}^{\infty}$ of $A^{(\alpha)}$ onto l^1, where $\hat{f}(m,n)=\int_{\bar{D}} f(z)R_{m,n}^{(\alpha)}(\bar{z})dm_\alpha(z)$. We now claim $A^{(\alpha)}$ is an algebra. Assume that $f(z)=\sum a_{m,n}R_{m,n}^{(\alpha)}(z)$ and $g(z)=\sum b_{k,l}R_{k,l}^{(\alpha)}(z)$ are in $A^{(\alpha)}$. Then we have

$$\begin{aligned} f(z)g(z) &= \sum_{m,n;k,l} a_{m,n}b_{k,l}R_{m,n}^{(\alpha)}(z)R_{k,l}^{(\alpha)}(z) \\ &= \sum_{p,q}\{\sum_{m,n;k,l} a_{m,n}b_{k,l}c_{p,q}(m,n\,;k,l)\}h_{p,q}^{(\alpha)}R_{p,q}^{(\alpha)}(z) \end{aligned}$$

and

$$\begin{aligned} \|fg\| &\leqq \sum_{p,q}\{\sum_{m,n;k,l} |a_{m,n}||b_{k,l}||\,c_{p,q}(m,n,;p,q)h_{p,q}^{(\alpha)}|\} \\ &\leqq \|f\|\|g\| \end{aligned}$$

since $\sum_{p,q}|c_{p,q}(m,n\,;k,l)h_{p,q}^{(\alpha)}|=1$ by (iv). Thus it follows that *the space $A^{(\alpha)}$ is a semi-simple, commutative Banach algebra with pointwise multiplication of functions.*

Let $\mathcal{M}$ be the maximal ideal space of $A^{(\alpha)}$. For every z in $\bar{D}$, the map $f\to f(z)$ defines a multiplicative linear functional on $A^{(\alpha)}$. Thus we have a map ι of $\bar{D}$ into $\mathcal{M}$ such that $\tilde{f}(\iota(z))=f(z)$ for z in $\bar{D}$ and f in $A^{(\alpha)}$, where $\tilde{f}$ is the Gelfand transform of f. It is clear that ι is one to one from $\bar{D}$ into $\mathcal{M}$. Moreover we can show that ι is a map of $\bar{D}$ onto $\mathcal{M}$ using an asymptotic formula for Jacobi polynomials $R_n^{(\alpha,\beta)}$ with error terms estimated with respect to the parameter β. Thus we have

Theorem 1. *The maximal ideal space $\mathcal{M}$ of the algebra $A^{(\alpha)}$ is homeomorphic to the closed unit disk $\bar{D}$ by the map ι and the Gelfand transform $\tilde{f}$ of f in $A^{(\alpha)}$ is given by $\tilde{f}(\iota(z))=f(z)$ for z in $\bar{D}$.*

By the Wiener-Lévy theorem we have

Corollary. *Suppose that $f(z)=\sum_{m,n}a_{m,n}R_{m,n}^{(\alpha)}(z)$, $\sum_{m,n}|a_{m,n}|<\infty$ and F is a holomorphic function on an open set containing the range of f. Then $F(f(z))=\sum_{m,n}b_{m,n}R_{m,n}^{(\alpha)}(z)$ with $\sum_{m,n}|b_{m,n}|<\infty$.*

Banach algebras related to some orthogonal polynomials are studied by several authors. For Jacobi polynomials, see G. Gasper [3] and S. Igari and Y. Uno [4]. A Banach algebra with the dual structure of $A^{(\alpha)}$ is studied by H. Annabi and K. Trimèche [1] and Y. Kanjin [6].

2. A closed set E in $\bar{D}$ will be called a *set of interpolation* with respect to $A^{(\alpha)}$, if every continuous function on E is the restriction of a function in $A^{(\alpha)}$ to E. S. A. Vinogradov [9] and [4] suggest the following observations.

A finite subset of $\bar{D}$ is evidently a set of interpolation with respect to $A^{(\alpha)}$. Let T be the circle group $R/2\pi Z$ and $A(T)$ be the space of absolutely convergent Fourier series $f(t)=\sum_{n=-\infty}^{\infty}a_ne^{int}$, $\sum_n|a_n|<\infty$. A closed set E in T is called a Helson set, if every continuous function on E is the restriction of a function in $A(T)$ to E (cf. [5; Ch. IV]).

The image of a Helson set by the map $t \to e^{it}$ will be called a Helson set in the boundary. For $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{int}$ in $A(T)$, put

$$f(z)=\sum_{n=0}^{\infty} a_n R_{n,0}^{(\alpha)}(z)+\sum_{n=1}^{\infty} a_{-n} R_{0,n}^{(\alpha)}(z).$$

Then $f(z)$ belongs to $A^{(\alpha)}$. Thus a Helson set in the boundary is a set of interpolation with respect to $A^{(\alpha)}$. Also, the union of a finite set in $\bar{D}$ and a Helson set in the boundary is a set of interpolation with respect to $A^{(\alpha)}$. The converse holds:

Theorem 2. *Suppose that $\alpha>1$. Then every set of interpolation with respect to $A^{(\alpha)}$ is the union of a finite set in D and a Helson set in the boundary.*

Remark. Whether Theorem 2 does hold or not for $1 \geqq \alpha \geqq 0$ is open. But we can show the following: Let $\alpha>0$ and E be a set of interpolation with respect to $A^{(\alpha)}$. Then points of E do not accumulate in D.

3. Let E be a closed subset of $\bar{D}$. Denote by $I(E)$ the closed ideal in $A^{(\alpha)}$ consisting of all f in $A^{(\alpha)}$ such that $f=0$ on E and by $J(E)$ the set of all f in $A^{(\alpha)}$ such that $f=0$ on a neighborhood of E. If $J(E)$ is dense in $I(E)$ then E is called a *set of spectral synthesis* for $A^{(\alpha)}$.

Theorem 3. *If $\alpha \geqq 1$ and z_0 is in D then $\{z_0\}$ is not a set of spectral synthesis for $A^{(\alpha)}$.*

We refer to [5; Ch. V] for the algebra $A(T)$ and F. Cazzaniga and C. Meaney [2] for the algebra of absolutely convergent Jacobi polynomial series. A proof of the theorems will be published elsewhere.

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