101. Continuability of Solutions of the Generalized Liénard System with Time Delay

By Jitsuro SUGIE
Department of Applied Physics, Osaka University

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1. Introduction. In this paper we consider the system of differential equations

$$(1.1)\qquad \begin{aligned} x'(t) &= y(t) - F(x(t)) \\ y'(t) &= -g(t, x(t-r(t))) \end{aligned}$$

where $x'(t)$ and $y'(t)$ denote the right-hand derivatives of x and y at t respectively, and $F: \boldsymbol{R} \to \boldsymbol{R}$, $g: [0, \infty) \times \boldsymbol{R} \to \boldsymbol{R}$, $r: [0, \infty) \to (0, \infty)$ are continuous. Note that other conditions on g, for example $xg(t, x) > 0$ if $x \neq 0$, are not assumed throughout this paper.

Following El'sgol'ts [2], for any $t_0 \geqq 0$, the initial interval at t_0 is given by $E_{t_0} = \{t_0\} \cup \{s : s = t - r(t) < t_0 \text{ for } t \geqq t_0\}$. For any $t_0 \geqq 0$ and any initial function $(\phi, \psi): E_{t_0} \to \boldsymbol{R}^2$, we say $(x(t), y(t))$ is a solution of (1.1) on $[t_0, T)$, where $t_0 < T \leqq \infty$, if $(x(t), y(t))$ is continuous on $E_{t_0} \cup [t_0, T)$ and satisfies (1.1) on (t_0, T) with $(x(t), y(t)) = (\phi(t), \psi(t))$ for all $t \in E_{t_0}$. We denote the solution by $(x(t; t_0, \phi, \psi), y(t; t_0, \phi, \psi))$.

For locally existence of solutions of delay-differential equations we refer the reader to Driver [1] or Hale [3].

The purpose of this paper is to give a necessary and sufficient condition for the continuability of solutions of (1.1).

In [4], Hara, Yoneyama and the author discussed continuation of solutions of the system without time delay

$$(1.2)\qquad \begin{aligned} x' &= y - F(x) \\ y' &= -g(x) \end{aligned}$$

and gave some necessary and sufficient conditions under which all solutions of (1.2) are continuable in the future. For example, the following result was given.

Theorem A. *Suppose that*

(i) $\quad xg(x) > 0$ *if* $|x| > k$ *for some* $k > 0$,

(ii) $\quad \sup_{x \geqq 0} F(x) < \infty$ *and* $\displaystyle\int_0^{\infty} \frac{g(x)}{1 + F_-(x)} dx < \infty$,

(iii) $\quad \inf_{x \leqq 0} F(x) > -\infty$ *and* $\displaystyle\int_0^{-\infty} \frac{g(x)}{1 + F_+(x)} dx < \infty$,

where $F_-(x) = \max\{0, -F(x)\}$ *and* $F_+(x) = \max\{0, F(x)\}$. *Then all solutions of* (1.2) *are continuable in the future if and only if*

$$\int_0^{\infty}\frac{dx}{1+F_-(x)}=\infty \quad and \quad \int_0^{-\infty}\frac{dx}{1+F_+(x)}=-\infty.$$

Theorem A suggests that the convergence and the divergence of the integrals $\int_0^{\infty}\frac{dx}{1+F_-(x)}$ and $\int_0^{-\infty}\frac{dx}{1+F_+(x)}$ play an important role on the continuability of solutions of (1.2). In this paper we show that the convergence and the divergence of the above integrals are also a valid criterion for the continuability of solutions of (1.1).

We make the following assumptions.

(A_1^+): $\int_0^{\infty}\frac{dx}{1+F_-(x)}=\infty,$

(A_2^+): there exists a sequence $\{x_n\}$ such that $x_n\to\infty$ and $F(x_n)\to\infty$ as $n\to\infty$,

(A_1^-): $\int_0^{-\infty}\frac{dx}{1+F_+(x)}=-\infty,$

and

(A_2^-): there exists a sequence $\{x_n\}$ such that $x_n\to-\infty$ and $F(x_n)\to-\infty$ as $n\to\infty$.

(A^+) is defined by (A_1^+) or (A_2^+). (A^-) is defined by (A_1^-) or (A_2^-). We now state our main result.

Theorem. *All solutions of* (1.1) *are continuable in the future if and only if* (A^+) *and* (A^-) *hold.*

Remark. Since $r(t)$ is a positive function, (1.1) is a delay-differential system in the strict sense and thus (1.1) does not include the ordinary differential system (1.2). Then the above Theorem and Theorem A are independent each other.

2. Preliminaries. We first give some well-known Lemmas which are useful to prove our result. Consider the ordinary differential equation

$$x'=f(x) \tag{2.1}$$

where f: $\boldsymbol{R}\to\boldsymbol{R}$ is continuous.

Lemma 1. *Suppose that there exists a sequence* $\{x_n\}$ *such that* $x_n\to\infty$ *as* $n\to\infty$ *and* $f(x_n)<0$ (*or* $x_n\to-\infty$ *as* $n\to\infty$ *and* $f(x_n)>0$). *Then all solutions of* (2.1) *are bounded from above* (*or below*) *as long as they exist.*

Lemma 2. *Suppose that* $f(x)>0$ *for all* $x\geqq 0$ *and* $\int_0^{\infty}\frac{dx}{f(x)}=\infty$ $\left(or\ f(x)<0\ for\ all\ x\leqq 0\ and \int_0^{-\infty}\frac{dx}{f(x)}=\infty\right)$. *Then all solutions of* (2.1) *are bounded from above* (*or below*) *on any bounded time interval.*

Lemma 3. *Suppose that* $f(x)>0$ *for all* $x\geqq 0$ *and* $\int_0^{\infty}\frac{dx}{f(x)}<\infty$.

Then for any $\tau>0$, there exists $x_0\geqq 0$ such that the maximal solution $x(t\,;0,x_0)$ of (2.1) tends to infinity as $t\to\tau^-$.

3. Proof of the theorem. We first prove the sufficiency. Suppose that there exist $t_0\geqq 0$, $T>t_0$, a continuous initial function (ϕ,ψ): $E_{t_0}\to \boldsymbol{R}^2$ and a solution $(x_1(t), y_1(t))=(x_1(t\,;t_0,\phi,\psi), y_1(t\,;t_0,\phi,\psi))$ of (1.1) such that $(x_1(t), y_1(t))$ is defined on $[t_0, T)$ and $\lim_{t\to T-}(x_1(t), y_1(t))$ does not exist. Since $r(T)>0$, then there exists $B>0$ such that $|x_1(t-r(t))|\leqq B$ for all $t\in[t_0, T]$. Let $L=\max_{t_0\leqq t\leqq T, |x|\leqq B}|g(t,x)|$ and $K=\max\{1, |\psi(t_0)|+L(T-t_0)\}$, then

$$|y_1'(t)|\leqq L \qquad \text{for all } t\in[t_0,T), \tag{3.1}$$

$$|y_1(t)|\leqq K \qquad \text{for all } t\in[t_0,T). \tag{3.2}$$

By (3.1), (3.2) and the continuity of $y_1(t)$ on $[t_0, T)$, there exists $y_1(T)=\lim_{t\to T-} y_1(t)$. Therefore we have

$$|y_1(t)|\leqq K \qquad \text{for all } t\in[t_0,T], \tag{3.3}$$

$$\lim_{t\to T-} x_1(t) \text{ does not exist.} \tag{3.4}$$

Consider the ordinary differential equation

$$x'=y_1(t)-F(x) \tag{3.5}$$

on $[t_0, T]\times \boldsymbol{R}$. Then it follows from (3.3) and $K\geqq 1$ that for all $t\in[t_0,T]$,

$$y_1(t)-F(x)\leqq K-F(x)\leqq K(1+F_-(x)), \tag{3.6}$$

$$y_1(t)-F(x)\geqq -K-F(x)\geqq -K(1+F_+(x)). \tag{3.7}$$

Now let us consider the equations

$$x'=K-F(x), \tag{3.8}$$

$$x'=K(1+F_-(x)), \tag{3.9}$$

$$x'=-K-F(x), \tag{3.10}$$

and

$$x'=-K(1+F_+(x)). \tag{3.11}$$

Then from (A^+) and Lemmas 1–2 we obtain that all solutions $x(t\,;t_0,\phi(t_0))$ of (3.8) or (3.9) are bounded from above on $[t_0, T]$ as long as they exist. Similarly, by (A^-) and Lemmas 1–2, all solutions $x(t\,;t_0,\phi(t_0))$ of (3.10) or (3.11) are bounded from below on $[t_0, T]$ as long as they exist. Therefore, using the comparison theorem, it follows from (3.6) and (3.7) that all solutions $x(t\,;t_0,\phi(t_0))$ of (3.5) are continuable up to $t=T$. On the other hand, $x_1(t)$ is a solution of (3.5) through $(t_0,\phi(t_0))$ on $[t_0, T)$ and satisfies (3.4). This is a contradiction.

We next prove the necessity. Suppose that all solutions of (1.1) are continuable in the future and

$$\int_0^{\infty}\frac{dx}{1+F_-(x)}<\infty \quad \text{and there exists } M>0 \text{ such that } F(x)\leqq M \text{ for all } x\geqq 0 \tag{3.12}$$

or

$$\int_0^{-\infty}\frac{dx}{1+F_+(x)}>-\infty \quad \text{and there exists } M>0 \tag{3.13}$$

such that $F(x) \geqq -M$ for all $x \leqq 0$.

We only consider the case (3.12), since the argument for the case (3.13) is similar.

Let $K > \max\{1, M\}$, $I = \{x \in \boldsymbol{R} : F(x) \geqq 0\}$ and $J = \{x \in \boldsymbol{R} : F(x) < 0\}$. Then we obtain

$$\begin{aligned}\int_0^\infty \frac{dx}{K-F(x)} &= \int_I \frac{dx}{K-F(x)} + \int_J \frac{dx}{K-F(x)} \\ &\leqq \frac{K}{K-M}\int_I \frac{dx}{K+F_-(x)} + \int_J \frac{dx}{K+F_-(x)} \\ &\leqq \frac{K}{K-M}\int_0^\infty \frac{dx}{1+F_-(x)} < \infty.\end{aligned}$$

Since $r(0) > 0$, there exists $\tau > 0$ such that

(3.14) $$t - r(t) \in E_0 \qquad \text{for all } t \in [0, \tau].$$

Therefore, by Lemma 3, there exists $x_0 > 0$ such that the maximal solution $x_2(t) = x_2(t; 0, x_0)$ of $x' = K - F(x)$ tends to infinity as $t \to \tau^-$, that is,

(3.15) $$x_2(t) \to \infty \quad \text{as} \quad t \to \tau^-.$$

Choose $y_0 \in \boldsymbol{R}$ such that $y_0 > K + N\tau$ where $N = \max_{0 \leqq t \leqq \tau} |g(t, x_0)|$. Let $\phi(s) = x_0$ and $\psi(s) = y_0$ for all $s \in E_0$, and consider a solution $(x_3(t), y_3(t)) = (x_3(t; 0, \phi, \psi), y_3(t; 0, \phi, \psi))$ of (1.1). Since $(x_3(t), y_3(t))$ is continuable in the future, then there exists $P > 0$ such that

(3.16) $$|x_3(t)| + |y_3(t)| \leqq P \qquad \text{for all } t \in [0, \tau].$$

It follows from (3.14) that $y_3'(t) = -g(t, x_0)$ for all $t \in [0, \tau]$, and hence $y_3(t) \geqq \psi(0) - N\tau > K$ for all $t \in [0, \tau]$. Therefore for all $t \in [0, \tau]$ we have $x_3'(t) = y_3(t) - F(x_3(t)) > K - F(x_3(t))$. Using the comparison theorem, we obtain from (3.16) that $x_2(t) \leqq x_3(t) \leqq P$ for all $t \in [0, \tau)$. This is a contradiction to (3.15). Thus the proof of Theorem is now complete.

References

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