11. *The Robin Problems on Domains with Many Tiny Holes*

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1. Introduction. Let Ω be a bounded domain of $\boldsymbol{R}^N$, $N\geqq 2$, with smooth boundary Γ and with the outer unit normal vector ν_Γ on Γ. Let $\boldsymbol{R}^N$ be divided into an infinitely many number of cubes C_i^ε, $i\in N$, with volume of $(2\varepsilon)^N$ and let $B_i(r^\varepsilon)$ be a closed ball of radius r^ε $(<\varepsilon)$ centered in C_i^ε. From Ω we remove all such balls and obtain a $D^\varepsilon(\subset\Omega)$ with n^ε holes. Under the case $n^\varepsilon\to\infty$, $r^\varepsilon\to 0$, a parameter, which determines the behavior of the Laplacian on D^ε with the Dirichlet condition, is known by M. Kac [2], J. Rauch and M. Taylor [3], D. Cioranescu and F. Murat [1]. That parameter is given by $\lim n^\varepsilon(r^\varepsilon)^{N-2}$ for $N\geqq 3$ and $\lim n^\varepsilon/|\log r^\varepsilon|$ for $N=2$, ε means values of a fixed sequence decreasing to zero. Now, we show a different parameter is important for the Robin problems.

From Ω we remove all balls $B_i(r^\varepsilon)$ such that dist $(B_i(r^\varepsilon),\Gamma)\geqq\varepsilon$ and obtain R^ε with n^ε holes. Let α be a positive constant and ν^ε the outer unit normal vector on ∂R^ε. We consider the Robin problem: for $f\in L^2(\Omega)$ find $u^\varepsilon\in H^1(R^\varepsilon)$ such that

$$(1)\qquad \begin{aligned} -\Delta u^\varepsilon&=f \qquad \text{a.e. in } R^\varepsilon,\\ \frac{\partial u^\varepsilon}{\partial\nu^\varepsilon}+\alpha u^\varepsilon&=0 \qquad \text{a.e. on } \partial R^\varepsilon.\end{aligned}$$

Theorem 1. *Let u^ε be the solution of* (1) *and $\tilde u^\varepsilon\in H^1(\Omega)$ an extension of u^ε to be harmonic in F^ε, $F^\varepsilon=\Omega\backslash R^\varepsilon$. Assume that $r^\varepsilon\to 0$ and $n^\varepsilon\to\infty$ with the conditions $\eta=\lim n^\varepsilon(r^\varepsilon)^{N-1}$, $0<\eta<\infty$. Then $\tilde u^\varepsilon$ converges weakly in $H^1(\Omega)$ to the solution of the problem:*

$$(2)\qquad \begin{aligned} -\Delta u+\frac{\alpha S_N\eta u}{|\Omega|}&=f \qquad \text{a.e. in } \Omega,\\ \frac{\partial u}{\partial\nu_\Gamma}+\alpha u&=0 \qquad \text{a.e. on } \Gamma.\end{aligned}$$

Here $|\Omega|$ means the volume of Ω and S_N means the surface area of the unit sphere of $\boldsymbol{R}^N$.

2. Abstract scheme. Let Ω be the same domain as in Section 1. We introduce a certain limit of the minus Laplacian, which corresponds to one of versions for theorem 1.2 of Cioranescu-Murat [1] in the case of Robin condition.

For a subdomain G of Ω we regard all functions of $L^2(G)$ as functions of $L^2(\Omega)$ vanishing outside G. In this section R^ε means a subdomain of Ω satisfying (a.1) below. Let $a^\varepsilon: H^1(R^\varepsilon)\times H^1(R^\varepsilon)\to\boldsymbol{R}$ be a bilinear form defined by

$$a^\varepsilon(v, w)=\int_{R^\varepsilon} \nabla v \cdot \nabla w dx+\alpha \int_{\partial R^\varepsilon} vw d\sigma.$$

Let us consider (1). For (1) we have

$$a^\varepsilon(u^\varepsilon, v)=\int_{R^\varepsilon} fv dx \qquad \text{for all } v \in H^1(R^\varepsilon). \tag{3}$$

We consider the behavior of the solution of (3). We use the norm of $H^1(R^\varepsilon)$ defined by $\|v\|^2_{H^1(R^\varepsilon)}=\|\nabla v\|^2_{L^2(R^\varepsilon)^N}+\|v\|^2_{L^2(R^\varepsilon)}$.

We set $F^\varepsilon=\Omega\backslash R^\varepsilon$. We assume the following conditions.

(a.1) F^ε does not meet Γ and its interior kernel is a nonempty set with locally Lipschitz boundary.

(a.2) There exists a family of extension maps $T^\varepsilon: H^1(R^\varepsilon)\to H^1(\Omega)$ such that

(i) $\limsup \|T^\varepsilon\|_{\mathcal{L}(H^1(R^\varepsilon), H^1(\Omega))}<\infty$,

(ii) if $\limsup a^\varepsilon(v^\varepsilon, v^\varepsilon)<\infty$ then $T^\varepsilon v^\varepsilon-v^\varepsilon\to 0$ in $L^2(\Omega)$.

(a.3) There exists a constant c_1 such that

$$a^\varepsilon(v, v)\geqq c_1\|v\|^2_{H^1(R^\varepsilon)} \text{ for all } v \in H^1(R^\varepsilon) \text{ and } \varepsilon>0.$$

(a.4) There exists a family $\{\theta^\varepsilon \in W^{1,\infty}(R^\varepsilon)\}_\varepsilon$ satisfying the conditions below.

(i) $\theta^\varepsilon=1$ a.e. on Γ.

(ii) Set $\tilde{\theta}^\varepsilon=T^\varepsilon\theta^\varepsilon$. Then $\tilde{\theta}^\varepsilon \xrightarrow{w} 1$ in $H^1(\Omega)$ as $\varepsilon\to 0$.

(iii) $\limsup a^\varepsilon(\theta^\varepsilon, \theta^\varepsilon)<\infty$.

(iv) There exists $\bar{\eta} \in W^{1,1}(\Omega)^*$ such that

$$-\nabla\cdot(\chi^\varepsilon\nabla\theta^\varepsilon)+\alpha\theta^\varepsilon\delta(\partial F^\varepsilon)\xrightarrow{s}\bar{\eta} \qquad \text{in } W^{1,1}(\Omega)^*,$$

$W^{1,1}(\Omega)^*$ means the dual space of $W^{1,1}(\Omega)$, χ^ε means the characteristic function of R^ε and $\delta(\partial F^\varepsilon)$ means the measure defined by

$$\langle\delta(\partial F^\varepsilon), v\rangle=\int_{\partial F^\varepsilon} v d\sigma \qquad \text{for all } v \in W^{1,1}(\Omega).$$

Theorem 2. *Let u^ε be the solution of* (3) *and set $\tilde{u}^\varepsilon=T^\varepsilon u^\varepsilon$. Under all the conditions from* (a.1) *to* (a.4), *$\tilde{u}^\varepsilon$ converges weakly in $H^1(\Omega)$ to u, where u is the solution of*

$$\begin{aligned} -\Delta u+\bar{\eta}u&=f \qquad \text{a.e. } in\ \Omega,\\ \frac{\partial u}{\partial \nu_\Gamma}+\alpha u&=0 \qquad \text{a.e. } on\ \Gamma. \end{aligned} \tag{4}$$

Proof. Substituting $v=u^\varepsilon$ into (3) and using (a.2) and (a.3) we can see

$$\limsup\|\tilde{u}^\varepsilon\|_{H^1(\Omega)}<\infty.$$

We can choose a weakly convergent subsequence $\{u_n\}_n$ such that $\tilde{u}_n\xrightarrow{w}u$ in $H^1(\Omega)$, where $u_n=u^{\varepsilon_n}$. It suffices to show

$$\int_\Omega \nabla u\cdot\nabla\zeta dx+\langle\bar{\eta}, u\zeta\rangle+\alpha\int_\Gamma u\zeta d\sigma=\int_\Omega f\zeta dx \tag{5}$$

for all $\zeta \in C^\infty(\bar{\Omega})$, where $\langle\ ,\ \rangle$ denotes the dual pair between $W^{1,1}(\Omega)^*$ and $W^{1,1}(\Omega)$. We can substitute $v=\theta_n\zeta$ into (3), where $\theta_n=\theta^{\varepsilon_n}$. We set $R_n=R^{\varepsilon_n}$ and $F_n=F^{\varepsilon_n}$. Thus

$$\begin{aligned} &\int_{R_n}\theta_n\nabla u_n\cdot\nabla\zeta dx+\int_{R_n}\zeta\nabla u_n\cdot\nabla\theta_n dx+\alpha\int_{\partial F_n}\zeta u_n\theta_n d\sigma+\alpha\int_\Gamma u_n\zeta d\sigma\\ &\quad=\int_{R_n} f\theta_n\zeta dx. \end{aligned} \tag{6}$$

Here we have used the equality $u_n = u_n\theta_n$ a.e. on Γ. By the definition of u_n and the conditions (a.2)-(ii), (a.4)-(ii) and (a.4)-(iii) we have $\nabla\tilde{u}_n \xrightarrow{w} \nabla u$, $\tilde{\theta}_n\nabla\zeta \xrightarrow{s} \nabla\zeta$ in $L^2(\Omega)^N$, $\tilde{\theta}_n - \theta_n \xrightarrow{s} 0$ in $L^2(\Omega)$ and $\tilde{\theta}_n \xrightarrow{s} 1$ in $L^2(\Omega)$, $\tilde{u}_n \longrightarrow u$ in $L^2(\Gamma)$. Thus,

$$\begin{aligned}\int_{R_n}\theta_n(\nabla u_n\cdot\nabla\zeta - f\zeta)\,dx + \alpha\int_\Gamma u_n\zeta d\sigma \\ = \int_\Omega \tilde{\theta}_n(\nabla\tilde{u}_n\cdot\nabla\zeta - f\zeta)dx + \alpha\int_\Gamma \tilde{u}_n\zeta d\sigma + \int_\Omega(\theta_n - \tilde{\theta}_n)[\nabla\tilde{u}_n\cdot\nabla\zeta - f\zeta]dx \\ \longrightarrow \int_\Omega(\nabla u\cdot\nabla\zeta - f\zeta)d\sigma + \alpha\int_\Gamma u\zeta d\sigma.\end{aligned}$$

Notice $\chi_n\zeta\nabla u_n = \chi_n[\nabla(\zeta\tilde{u}_n) - \tilde{u}_n\nabla\zeta]$, where $\chi_n = \chi^{\varepsilon_n}$. Therefore,

$$\begin{aligned}\int_{R_n}\zeta\nabla u_n\cdot\nabla\theta_n dx + \alpha\int_{\partial F_n} u_n\theta_n\zeta d\sigma \\ = \langle -\nabla\cdot(\chi_n\nabla\tilde{\theta}_n) + \alpha\theta_n\delta(\partial F_n),\ \tilde{u}_n\zeta\rangle - \int_{R_n} u_n\nabla\theta_n\cdot\nabla\zeta dx.\end{aligned}$$

Using (a.2)-(ii) and (a.4)-(ii) we have $u_n\nabla\zeta \xrightarrow{s} u\nabla\zeta$, $(u_n - \tilde{u}_n)\nabla\zeta \xrightarrow{s} 0$ in $L^2(\Omega)^N$, $\nabla\tilde{\theta}_n \xrightarrow{w} 0$ in $L^2(\Omega)^N$. Clearly, $\zeta\tilde{u}_n \xrightarrow{w} \zeta u$ in $W^{1,1}(\Omega)$. Also, using (a.4)-(iv) we get

$$\int_{R_n} u_n\nabla\theta_n\cdot\nabla\zeta dx = \int_\Omega \tilde{u}_n\nabla\tilde{\theta}_n\cdot\nabla\zeta dx + \int_\Omega (u_n - \tilde{u}_n)\nabla\tilde{\theta}_n\cdot\nabla\zeta dx \longrightarrow 0,$$

$$\int_{R_n}\zeta\nabla u_n\cdot\nabla\theta_n dx + \alpha\int_{\partial F_n} u_n\theta_n\zeta d\sigma \longrightarrow \langle\bar{\eta}, u\zeta\rangle.$$

Therefore we get (5). The proof is completed. q.e.d.

3. Proof of Theorem 1. We introduce special functions θ^ε such that $\theta^\varepsilon = 1$ on $\Omega\setminus\cup\{B_i(\varepsilon): 1\leqq i\leqq n^\varepsilon\}$, $\Delta\theta^\varepsilon = 0$ on $\cup\{B_i(\varepsilon)\setminus B_i(r^\varepsilon): 1\leqq i\leqq n^\varepsilon\}$ and $\partial\theta^\varepsilon/\partial\nu^\varepsilon + \alpha\theta^\varepsilon = 0$ on $\partial F^\varepsilon(\subset\partial R^\varepsilon)$. By a similar way to that in [1] we can see functions θ^ε satisfies (a.4) with $\bar{\eta} = \alpha S_N\eta/|\Omega|$. We show (a.3). We set $E_i = C_i^\varepsilon\cap R^\varepsilon$, $1\leqq i\leqq n^\varepsilon$. Choose ε so small that

$$2\int_{r^\varepsilon}^{N^{1/2}\varepsilon} r^{N-1}dr\max\left\{\int_{r^\varepsilon}^{N^{1/2}\varepsilon}\rho^{1-N}d\rho,\ (r^\varepsilon)^{1-N}\right\} \leqq \frac{3}{\eta}\left(\frac{N}{2}\right)^N|\Omega|.$$

Here we use $n^\varepsilon \sim |\Omega|/(2\varepsilon)^N$, $\varepsilon\to 0$. For the equality,

$$|v(r,\omega)|^2 = \left[\int_{r^\varepsilon}^r \frac{\partial v}{\partial\rho}d\rho + v(r^\varepsilon,\omega)\right]^2,\quad v\in H^1(R^\varepsilon),\quad \omega\in\partial B_i(1),$$

using the Schwarz inequalities twice on the left hand side, multiplying both sides by r^{N-1}, integrating them over $(r^\varepsilon, \rho_i(\omega))$, $\rho_i(\omega) = \sup\{r\geqq 0: r\omega\in E_i\}$, next, over the unit sphere $\partial B_i(1)$, we get the inequality

$$\int_{E_i}|v|^2dx \leqq 3\left(\frac{N}{2}\right)^N\frac{|\Omega|}{\eta}\left[\int_{E_i}|\nabla v|^2dx + \int_{\partial B_i(r^\varepsilon)}|v|^2d\sigma\right].$$

We denote by G_i a non-empty set $C_i^\varepsilon\cap R^\varepsilon$, $i > n^\varepsilon$. For sufficiently small ε we have a Lipschitz function h_i such that $G_i = \{(x', x_N): |x_i|\leqq\varepsilon, 1\leqq i\leqq N-1, 0\leqq x_N\leqq h_i(x')\}$ and $C_i^\varepsilon\cap\Gamma = \{(x', h_i(x'): |x_i|\leqq\varepsilon, 1\leqq i\leqq N-1\}$, $x' = (x_1, \cdots, x_{N-1})$. Similarly, for the same v as the above we have a constant C_1 independent of ε such that

$$\int_{G_i} |v|^2 dx \leqq C_1 \varepsilon \left(\int_{G_i} |\nabla v|^2 dx + \int_{C_i^\varepsilon \cap \Gamma} |v|^2 d\sigma \right).$$

By these inequalities we can see (a.3). The family of extensions: $H^1(R^\varepsilon) \to H^1(\Omega)$ in Theorem 1 is uniformly bounded with respect to their operator norm (cf. [3]). So (a.2)-(i) holds. The condition (a.1) also holds. Lastly, (a.2)-(ii) follows from the fact such that the linear operator: $L^2(\partial B_i(r^\varepsilon)) \ni v \to v_r \in L^2(\partial B_i(r^\varepsilon))$ is bounded with norm one for $0<r<1$. Here $v_r(r^\varepsilon, \omega) = \tilde{v}(x_i^\varepsilon + r r^\varepsilon \omega)$, where x_i^ε denotes the center of $B_i(r^\varepsilon)$. The conditions of Theorem 2 are all verified. q.e.d.

References

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