

## 60. On the Diffeomorphism Types of Elliptic Surfaces

By Masaaki UE

Department of Mathematics, Faculty of Science, University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 12, 1985)

An *elliptic surface* [3] is a complex surface  $M$  with a holomorphic map  $\pi$  of  $M$  onto a Riemann surface  $S$  such that the inverse image  $\pi^{-1}(p)$  of any general point  $p$  is an elliptic curve. Matsumoto [4] [5] proved that the diffeomorphism type of  $M$  is completely determined by its euler number  $e(M)$  and the genus of  $S$  if  $M$  contains no multiple fiber. (The case when the genus of  $S$  is 0 was proved by Kas [2] and Moishezon [6].) The case when  $M$  has multiple fibers is more difficult and actually there are examples with exotic smooth structures (Dolgacev surfaces) as was proved by Donaldson [1], Morgan, and Friedman. However we can show that in many cases the diffeomorphism types of the elliptic surfaces are completely determined by their euler numbers and their fundamental groups. By Moishezon [6] we may assume that every singular fiber of the elliptic surfaces with which we are concerned is either a multiple torus  ${}_mI_0$  or a fiber of type  $I_1$  ([3]). Let  $\pi: M \rightarrow S$  be such an elliptic surface. We can consider  $S$  as a 2-orbifold such that every point  $p_i$  which is the image by  $\pi$  of a multiple torus of multiplicity  $m_i$  is a cone point of cone angle  $2\pi/m_i$  ( $i=1, \dots, k$ ). Then we have:

**Theorem.** *Let  $\pi: M \rightarrow S$  and  $\pi': M' \rightarrow S'$  be the relatively minimal elliptic surfaces. Suppose that  $S$  and  $S'$  are either euclidean or hyperbolic. Then  $M$  is diffeomorphic to  $M'$  if and only if  $e(M)=e(M')$  and  $\pi_1 M \cong \pi_1 M'$ .*

This theorem is divided into the following two cases.

**Case 1.**  $e(M) (e(M')) > 0$ . This implies that  $M(M')$  contains at least one singular fiber other than a multiple torus. In this case Theorem also holds when  $S(S')$  is spherical with 3 cone points and is derived from:

**Claim A.** *If  $S$  is isomorphic to  $S'$  as 2-orbifolds, then  $M$  is diffeomorphic to  $M'$  if and only if  $e(M)=e(M')$ .*

**Claim B.** *If  $S$  is not isomorphic to  $S'$ , then  $\pi_1 M \neq \pi_1 M'$ .*

**Case 2.**  $e(M)=e(M')=0$ . In this case every singular fiber of  $M(M')$  is a multiple torus.  $M$  and  $M'$  are considered as 4-dimensional Seifert fiberings studied by Thornton [8] and Zieshang [9]. Theorem in this case was proved by Zieshang [9] if  $S$  and  $S'$  are hyperbolic, and was proved by Sakamoto-Fukuhara [7] if  $S=S'=T^2$ . In the other cases we can see that  $\pi_1 M \cong \pi_1 M'$  implies that there is a diffeomorphism between  $M$  and  $M'$  (not necessarily fiber-preserving). We can also see that there are seven examples each of which admits both the structure of a  $T^2$ -bundle over  $T^2$  and

the structure of a Seifert fibering over a euclidean 2-orbifold of genus 0. The details of the proof will appear elsewhere.

### References

- [1] S. K. Donaldson: The differential topology of complex surfaces. Oxford (1985) (preprint).
- [2] A. Kas: On the deformation types of regular elliptic surfaces. Complex Analysis and Algebraic Geometry. Iwanami and Cambridge Univ. Press, pp.107–112 (1977).
- [3] K. Kodaira: On compact analytic surfaces II. Ann. of Math., **177**, 563–626 (1963).
- [4] Y. Matsumoto: Diffeomorphism types of elliptic surfaces. Proc. Japan Acad., **61A**, 55–58 (1985).
- [5] —: Diffeomorphism types of elliptic surfaces. Univ. of Tokyo (1985) (preprint).
- [6] B. Moishezon: Complex surfaces and connected sums of complex projective planes. Lect. Note in Math., Springer, vol. 603, Berlin-Heidelberg-New York (1977).
- [7] K. Sakamoto and S. Fukuhara: Classification of  $T^2$ -bundles over  $T^2$ . Tokyo J. Math., **6**, no. 2, 311–328 (1983).
- [8] M. C. Thornton: Singularly fibered manifolds. Illinois J. Math., **11**, 189–201 (1967).
- [9] H. Zieshang: On toric fiberings over surfaces. Math. Zametki, **5**, 341–345 (1969).