55. A Note on Jacobi's Generating Function for the Jacobi Polynomials

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Some elementary identities in the theory of the Gaussian hypergeometric series are used here to present a simple proof of Jacobi's generating function for the Jacobi polynomials.

In the literature there are several interesting proofs of Jacobi's generating function for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta}, \tag{1}$$

where $R=(1-2xt+t^2)^{1/2}$. See, for example, Szegö [6, Section 4.4], Rainville [4, Section 140], Carlitz [2], Askey [1], and Foata and Leroux [3]; see also Srivastava and Manocha [5, p. 82]. We give here a simple proof which uses the definition [6, p. 62, Equation (4.21.2)]

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \begin{bmatrix}\alpha+n\\ n\end{bmatrix} {}_2F_1\begin{bmatrix} -n,\ \alpha+\beta+n+1; & \\ & \dfrac{1-x}{2} \\ \alpha+1; & \end{bmatrix} \\ &= \sum_{k=0}^{n} \begin{bmatrix}\alpha+n\\ n-k\end{bmatrix} \begin{bmatrix}\alpha+\beta+n+k\\ k\end{bmatrix} \left[\frac{x-1}{2}\right]^k, \end{aligned} \tag{2}$$

and such elementary results from the theory of the Gaussian hypergeometric series ${}_2F_1$ as the transformation [4, p. 60, Equation (4)]

$${}_2F_1\begin{bmatrix} a,\ b; & \\ & z \\ c; & \end{bmatrix} = (1-z)^{-a}\,{}_2F_1\begin{bmatrix} a,\ c-b; & \\ & -\dfrac{z}{1-z} \\ c; & \end{bmatrix}, \tag{3}$$

the reduction formula [4, p. 70, Problem 10]

$${}_2F_1\begin{bmatrix} a,\ a+\dfrac{1}{2}; & \\ & z \\ 2a; & \end{bmatrix} = \frac{1}{\sqrt{1-z}}\left[\frac{1+\sqrt{1-z}}{2}\right]^{1-2a}, \tag{4}$$

and the binomial expansion [4, p. 58, Equation (1)]

$${}_2F_1\begin{bmatrix} a,\ b; & \\ & z \\ b; & \end{bmatrix} = \sum_{n=0}^{\infty}\begin{bmatrix}a+n-1\\ n\end{bmatrix} z^n = (1-z)^{-a}. \tag{5}$$

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Denoting the left-hand member of (1) by $G(x, t)$, and using the definition (2), we readily have

$$(6)\qquad G(x,t)=\sum_{k=0}^{\infty}\begin{bmatrix}\alpha+\beta+2k\\ k\end{bmatrix}\left\{\frac{1}{2}(x-1)t\right\}^{k}{}_{2}F_{1}\begin{bmatrix}\alpha+\beta+2k+1,\ \alpha+k+1;\\ & t\\ \alpha+\beta+k+1;\end{bmatrix}$$

$$=(1-t)^{-\alpha-\beta-1}\sum_{k=0}^{\infty}\begin{bmatrix}\alpha+\beta+2k\\ k\end{bmatrix}\left\{\frac{(x-1)t}{2(1-t)^{2}}\right\}^{k}$$

$$\cdot\,{}_{2}F_{1}\begin{bmatrix}\alpha+\beta+2k+1,\ \beta;\\ & -\dfrac{t}{1-t}\\ \alpha+\beta+k+1;\end{bmatrix},$$

where we have employed the transformation (3).

Upon rewriting this last expression in (6), it is easily observed that

$$(7)\qquad G(x,t)=(1-t)^{-\alpha-\beta-1}\sum_{n=0}^{\infty}\begin{bmatrix}\beta+n-1\\ n\end{bmatrix}\left[-\frac{t}{1-t}\right]^{n}$$

$$\cdot\,{}_{2}F_{1}\begin{bmatrix}\frac{1}{2}(\alpha+\beta+n+1),\ \frac{1}{2}(\alpha+\beta+n+2);\\ & \dfrac{2(x-1)t}{(1-t)^{2}}\\ \alpha+\beta+n+1;\end{bmatrix}.$$

Now apply the reduction formula (4) with

$$(7)\qquad a=\frac{1}{2}(\alpha+\beta+n+1)\quad\text{and}\quad z=\frac{2(x-1)t}{(1-t)^{2}},$$

and we find from (7) that

$$(9)\qquad G(x,t)=2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha-\beta}\sum_{n=0}^{\infty}\begin{bmatrix}\beta+n-1\\ n\end{bmatrix}\left[-\frac{2t}{1-t+R}\right]^{n},$$

where R is defined as in (1).

Finally, the generating function (1) follows at once from (9) by appealing to the elementary identity (5), and our proof is thus completed.

References

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