53. *On Strong Hyperbolicity for First Order Systems*

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§1. **Introduction.** In this note, we shall study strong hyperbolicity for first order hyperbolic systems;

$$L(x, D)=-D_0+\sum_{j=1}^{d} A_j(x)D_j+B(x),$$

where $A_j(x)$, $B(x)$ are $N\times N$ matrices with smooth entries defined near the origin in R^{d+1} with coordinates $x=(x_0, x')=(x_0, x_1, \cdots, x_d)$ and $D_j=-i(\partial/\partial x_j)$. Denote $\xi=(\xi_0, \xi')=(\xi_0, \xi_1, \cdots, \xi_d)$ and by $h(x, \xi)$ the determinant of the principal symbol $L_1(x, \xi)$ of $L(x, D)$;

$$L_1(x, \xi)=-\xi_0+\sum_{j=1}^{d} A_j(x)\xi_j,$$

and say that $L_1(x, \xi)$ is strongly hyperbolic if the Cauchy problem for $L(x, D)$ is C^∞ well posed near the origin for any lower order term $B(x)$ ([8]). Throughout this paper, we assume that $h(x, \xi)$ is hyperbolic with respect to dx_0 near the origin, i.e. $h(x, \xi_0, \xi')=0$ has only real roots for any (x, ξ'), $\xi'\in R^d\backslash 0$, $x\in R^{d+1}$ (x near the origin) and furthermore we assume that the multiplicities of these characteristic roots are at most two.

We shall prove that if $L_1(x, \xi)$ is strongly hyperbolic near the origin then at every point $(x, \xi)\in T^*R^{d+1}\backslash 0$ (x near the origin), $L_1(x, \xi)$ is effectively hyperbolic or diagonalizable (that is similar to a diagonal matrix). Conversely when $L_1(x, \xi)$ is effectively hyperbolic at every $\rho=(\bar{x}, \bar{\xi})$ with $\pi(\rho)=(\bar{x}, \bar{\xi}')$, we know that for any $B(x)$, there is a parametrix of $L(x, D)$ near $(\bar{x}', \bar{\xi}')$ with finite propagation speed of wave front sets ([10]), where π is the projection from T^*R^{d+1} to $R\times T^*R^d$ off ξ_0. In case $L_1(x, \xi)$ is diagonalizable near every ρ with $\pi(\rho)=(\bar{x}, \bar{\xi}')$, we shall show, under some additional conditions, that $L_1(x, \xi)$ is smoothly symmetrizable near $(\bar{x}, \bar{\xi}')$. Hence for any $B(x)$, $L(x, D)$ has a parametrix near $(\bar{x}', \bar{\xi}')$ with finite propagation speed of wave front sets.

§2. **Notations and results.** Let $L_0(x, \xi)$ be the symbol of degree 0 of $L(x, \xi)$, ${}^{co}L_1(x, \xi)$ the cofactor matrix of $L_1(x, \xi)$, and $L^s(x, \xi)$ the subprincipal symbol of $L(x, \xi)$;

$$L^s(x, \xi)=L_0(x, \xi)+\frac{i}{2}\sum_{j=0}^{d}(\partial^2/\partial\xi_j\partial x_j)L_1(x, \xi).$$

We denote by $F(\rho)$ the Fundamental (Hamilton) matrix corresponding to the Hessian Q of $h/2$ at ρ and set

$$Tr^+h(\rho)=\sum \mu_j$$

where $i\mu_j$ are the eigenvalues of $F(\rho)$ on the positive imaginary axis repeated according to their multiplicities. We say that $L_1(x,\xi)$ is effectively hyperbolic at ρ if $F(\rho)$ has non zero real eigenvalues (see [4], [5]).

We introduce the following symbol (see [2]).

$$l(x,\xi)=L^s(x,\xi)^{co}L_1(x,\xi)-\frac{i}{2}\left\{L_1, {}^{co}L_1\right\}(x,\xi),$$

where $\{L_1, {}^{co}L_1\}=\sum_{j=0}^{d}((\partial/\partial\xi_j)L_1(\partial/\partial x_j)({}^{co}L_1)-(\partial/\partial x_j)L_1(\partial/\partial\xi_j)({}^{co}L_1))$. Let $\rho=(\bar{x},\bar{\xi})\in T^*R^{d+1}\backslash 0$ be a point such that $dh(\rho)=0$. We say that $L_1(x,\xi)$ satisfies the condition (H) at ρ if there are a real number α with $|\alpha|\leqq 1$ and $N\times N$ matrix Ω such that

(H) $$l(\rho)+\alpha Tr^+h(\rho)=L_1(\rho)\Omega.$$

This condition corresponds to the Ivrii-Petkov condition for scalar non effectively hyperbolic operators ([4], [5]). We note that the condition (H) remains invariant after similarity transformations to L by non singular smooth matrix and also invariant after changes of coordinates x. These facts follow from the proof of Theorem 1 in [11] (see also [1]).

First we state a necessary condition for C^∞ well posedness of the Cauchy problem for $L(x,D)$.

Theorem 2.1. *Let $\rho\in T^*R^{d+1}\backslash 0$. We assume that $L_1(x,\xi)$ is not effectively hyperbolic nor diagonalizable at ρ. Then for the Cauchy problem for $L(x,D)$ to be well posed, the condition* (H) *at ρ is necessary.*

In case $F(\rho)$ is nilpotent, the condition (H) yields to the Levi condition. From this theorem it follows that

Corollary 2.1. *Suppose that $L_1(x,\xi)$ is strongly hyperbolic near the origin. Then at every $(x,\xi)\in T^*R^{d+1}\backslash 0$ (x near the origin), $L_1(x,\xi)$ is effectively hyperbolic or diagonalizable.*

Remark 2.1. In case $d=1$, this result was shown in [9] and the case when $A_j(x)$ depend only on x_0, this is obtained in [14]. In both cases we have $Tr^+h(\rho)=0$.

Theorem 2.2 ([10]). *Assume that $L_1(x,\xi)$ is effectively hyperbolic at every ρ such that $\pi(\rho)=(\bar{x},\bar{\xi}')$. Then, in a sufficiently small conic neighborhood of $(\bar{x}',\bar{\xi}')$, there is a parametrix of $L(x,D)$ with finite propagation speed of wave front sets.*

Remark 2.2. In general, effectively hyperbolic systems are not smoothly symmetrizable because effectively hyperbolic systems may cause a great loss of regularity of solutions in contrast to symmetric systems.

Next, taking Corollary 2.1 and Theorem 2.2 into account, we study the case that $L_1(x,\xi)$ is diagonalizable near every ρ such that $\pi(\rho)=(\bar{x},\bar{\xi}')$. In this case we impose the same condition on the double characteristic set of h under which the Cauchy problem for scalar non effectively hyperbolic operators was studied in [4] and [6].

(2.3) The double characteristic set $\Sigma=\{(x,\xi)\,;\,dh(x,\xi)=0\}$ is a manifold and the rank of the Hessian of h is equal to the codimension of Σ at every point of Σ.

Lemma 2.1. *Assume* (2.3) *and that* $L_1(x, \xi)$ *is diagonalizable near every* ρ *with* $\pi(\rho)=(\bar{x}, \bar{\xi}')$. *Then the codimension of* Σ *is at most* 4. *Moreover if* $L_1(x, \xi)$ *is real then the codimension of* Σ *is at most* 3.

Theorem 2.3. *Assume* (2.3) *and that* $L_1(x, \xi)$ *is diagonalizable near every* ρ *with* $\pi(\rho)=(\bar{x}, \bar{\xi}')$ *and the codimension of* Σ *is at most* 3. *Then* $L_1(x, \xi)$ *is smoothly symmetrizable near* $(\bar{x}, \bar{\xi}')$ i.e. *there is a smooth matrix* $S(x, \xi')$, *homogeneous of degree* 0 *in* ξ' *defined near* $(\bar{x}, \bar{\xi}')$ *and satisfying the following conditions,*

(2.4) $$S(x, \xi')=S^*(x, \xi')>0$$

(2.5) $$S(x, \xi')L_1(x, \xi)=L_1^*(x, \xi)S(x, \xi'),$$

where a^* *denotes the adjoint matrix of* a.

Corollary 2.2. *Assume* (2.3) *and that* $L_1(x, \xi)$ *is real and diagonalizable near every* ρ *with* $\pi(\rho)=(\bar{x}, \bar{\xi}')$. *Then the same conclusion as Theorem* 2.3 *holds with real symmetric* S.

Remark 2.3. When the characteristic roots are of constant multiplicities, $L_1(x, \xi)$ is strongly hyperbolic if and only if $L_1(x, \xi)$ is diagonalizable ([7]). This case occurs only when the codimension of Σ is 1.

Theorem 2.4. *Assume* (2.3) *and that* $L_1(x, \xi)$ *is real and at every point* $\rho \in \Sigma$ *with* $\pi(\rho)=(\bar{x}, \bar{\xi}')$, *one of the following conditions is fulfilled,*

(2.6) $L_1(x, \xi)$ *is effectively hyperbolic at* ρ,

(2.7) $L_1(x, \xi)$ *is diagonalizable near* ρ *on* Σ.

Then, in a sufficiently small conic neighborhood of $(\bar{x}', \bar{\xi}')$, $L_1(x, \xi)$ *has a parametrix with finite propagation speed of wave front sets for any* $B(x)$.

§3. Remarks on localizations. In this section we assume that $N=2$ and $L_1(x, \xi)$ is diagonalizable at ρ where $dh(\rho)=0$. Denote by $L_\rho(x, \xi)$ the lowest homogeneous part in the Taylor expansion of $L_1(x, \xi)$ at ρ. Then it is clear that $L_1(\rho)=0$ and hence $L_\rho(x, \xi)$ is a first order system in (x, ξ), which may be called the localization of $L_1(x, \xi)$ at ρ (cf. [13]).

Proposition 3.1. *Let* $N=2$ *and* $L_1(\rho)=0$. *Then the rank of the Hessian of* h *at* ρ *is at most* 4 (resp. 3 *in the case* $A_j(x)$ *are real*). *If the rank of the Hessian of* h *at* ρ *is* 4 (resp. 3 *in the case* $A_j(x)$ *are real*) *then with some non singular matrix* T, $T^{-1}L_\rho(x, \xi)T$ *is Hermitian for all* $(x, \xi) \in R^{2d+2}$. *Hence* $L_\rho(x, \xi)$ *is strongly hyperbolic system on* $T(T^*R^{d+1})$ *with respect to* dx_0.

Proposition 3.2. *Let* $N=2$ *and assume that* (2.3) *and* $L_1(x, \xi)$ *is diagonalizable at every point of* Σ *near* ρ. *Then the same conclusion as in Proposition* 3.2 *holds.*

Remark 3.1. In constant coefficients case, (2.3) is always fulfilled, then from Proposition 3.2, we reobtain Theorem 5 in [12] since $L_\rho(x, \xi)=L_1(\xi)$.

Remark 3.2. Proposition 3.2 is an analogue of the result of [13] for strongly hyperbolic systems with constant coefficients. However, in general, strong hyperbolicity of $L_\rho(x, \xi)$ is not necessary for strong hyperbolicity of $L_1(x, \xi)$ even if $L_1(\rho)=0$.

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