96. *Infinitely Many Periodic Solutions for a Superlinear Forced Wave Equation*

By Kazunaga TANAKA
Department of Mathematics, Waseda University

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1985)

1. Introduction. In this article we shall study the nonlinear wave equation:

(1) $v_{tt}-v_{xx}+g(v)=f(x,t), \quad (x,t)\in(0,\pi)\times\boldsymbol{R},$

(2) $v(0,t)=v(\pi,t)=0, \quad t\in\boldsymbol{R},$

(3) $v(x,t+2\pi)=v(x,t), \quad (x,t)\in(0,\pi)\times\boldsymbol{R},$

where $g\in C(\boldsymbol{R},\boldsymbol{R})$ is a function such that $g(\xi)/\xi\to\infty$ as $|\xi|\to\infty$ and $f(x,t)$ is a 2π-periodic function of t.

In a previous paper K. Tanaka [5] we studied (1)–(3) in case $g(\xi)=\pm|\xi|^{s-1}\xi$. This paper is a continuation of [5] and deals with more general equations. Our main result is as follows:

Theorem. *Suppose that* $g\in C(\boldsymbol{R},\boldsymbol{R})$ *satisfies*

(g_1) $g(\xi)$ *is strictly increasing,*

(g_2) *there exist* $\mu>2$ *and* $l\geqq 0$ *such that for* $|\xi|\geqq l$,

$$0<\mu G(\xi)\equiv\mu\int_0^\xi g(\tau)d\tau\leqq\xi g(\xi),$$

(g_3) *there exist* $s>1$ *and* $C>0$ *such that for* $\xi\in\boldsymbol{R}$,

$$|g(\xi)|\leqq C(|\xi|^s+1),$$

(g_4) $\dfrac{2}{s-1}>\dfrac{\mu}{\mu-1}.$

Then, for all 2π*-periodic* $f(x,t)\in L^\infty([0,\pi]\times\boldsymbol{R})$, *there exists an unbounded sequence of weak solutions of* (1)–(3) *in* L^∞.

In [3], P. H. Rabinowitz obtained the conditions which ensure the existence of an unbounded sequence of solutions of the semilinear elliptic equation:

$$-\Delta u=g(u)+f(x), \quad x\in D,$$
$$u=0, \quad x\in\partial D,$$

where $D\subset\boldsymbol{R}^n$ is a smooth bounded domain. In particular, in case $n=2$, his conditions are (g_2), (g_3), (g_4) and

(g_5) $g(-\xi)=-g(\xi)$ for all $\xi\in\boldsymbol{R}$.

He also obtained a similar existence result for the second order Hamiltonian systems of ordinary differential equations. For the wave equation (1)–(3), we act on S^1-symmetry and get the existence result without assumption (g_5).

As in K. Tanaka [5], we use a perturbation result of P. H. Rabinowitz [3] asserting the existence of infinitely many critical points of perturbed symmetric functionals and the dual variational formulation of the problem (1)–(3). Details of the proof will be published elsewhere.

2. Outline of the proof. Let $\Omega=(0,\pi)\times(0,2\pi)$ and $h(\xi)=$the inverse function of $g(\xi)$. Set

$$q=\frac{\mu}{\mu-1}\in(1,2) \quad\text{and}\quad r=\frac{1}{s}+1.$$

Consider the operator $Au=u_{tt}-u_{xx}$ acting on functions in $L^1(\Omega)$ satisfying (2) and (3). Denote by N the kernel of A. We act on the space

$$E=\left\{u\in L^q(\Omega)\,;\ \int_\Omega u\phi=0\ \text{ for all } \phi\in N\cap L^\mu(\Omega)\right\}$$

with L^q norm $\|\cdot\|_q$. For $\theta\in[0,2\pi)\simeq S^1$, define $T_\theta: E\to E$ by $(T_\theta u)(x,t)=u(x,t+\theta)$.

For any $u\in E$ there exists a unique $Ku\in E$ such that $A(Ku)=u$. Moreover the operator $K: E\to E^*$ is compact.

We define the functional $I(u)\in C^1(E,\boldsymbol{R})$ by

$$I(u)=\frac{1}{2}\int_\Omega (Ku)u+\int_\Omega H(u+f),$$

where $H(\xi)=\int_0^\xi h(\tau)\,d\tau$. There is a one-to-one correspondence between the critical points of $I(u)$ and the weak solutions of (1)–(3).

To verify the Palais-Smale compactness condition, we replace $I(u)$ by $I(\varepsilon\,;u)\in C^1(E,\boldsymbol{R})$ $(\varepsilon\in[0,1])$ defined by

$$I(\varepsilon\,;u)=\frac{1}{2}\int_\Omega (Ku)u+\int_\Omega H(u+f)+\int_\Omega \omega(\varepsilon u),$$

where $\omega\in C^2(\boldsymbol{R},\boldsymbol{R})$ is an even convex function such that $\omega(\xi)=|\xi|^q$ for $|\xi|\geqq 1$, $\omega(\xi)=0$ for $|\xi|\leqq c_q$, where $c_q>0$ is a constant. Then $I(\varepsilon\,;u)$ satisfy the Palais-Smale condition for all $\varepsilon\in(0,1]$.

As in K. Tanaka [5], we use another modified functional $J(\varepsilon\,;u)\in C^1(E,\boldsymbol{R})$ defined by

$$J(\varepsilon\,;u)=\frac{1}{2}\int_\Omega (Ku)u+\int_\Omega H(u)+\int_\Omega \omega(\varepsilon u)+\psi(\varepsilon\,;u)\int_\Omega (H(u+f)-H(u)),$$

where $\psi(\varepsilon\,;u)$ will be defined analogously as in K. Tanaka [5]. Here we can assume that $J(\varepsilon\,;u)$ is a nondecreasing function of $\varepsilon\in[0,1]$ for fixed $u\in E$. In what follows we denote by "$'$" the Fréchet derivative with respect to u.

Lemma 1. *There is a constant $M>0$ independent of $\varepsilon\in(0,1]$ such that*

(i) *$J(\varepsilon\,;u)$ satisfies the Palais-Smale condition on*

$$\hat{A}_M(\varepsilon)=\{u\in E\,;\ J(\varepsilon\,;u)\geqq M\}.$$

(ii) *$J(\varepsilon\,;u)\geqq M$ and $J'(\varepsilon\,;u)=0$ imply that $J(\varepsilon\,;u)=I(\varepsilon\,;u)$ and $I'(\varepsilon\,;u)=0$.*

Note that K is a compact self-adjoint operator in $E\cap L^2(\Omega)$. Its eigenvalues are $\{1/(j^2-k^2)\,;\ j\neq k\}$. We rearrange the negative eigenvalues in the following order, denoted by

$$-\mu_1\leqq-\mu_2\leqq-\mu_3\leqq\cdots<0.$$

Here, for each n, there is a one-to-one correspondence between μ_n and a 2-dimensional invariant subspace:

$$\text{Span}\{e_n^+=\sin jx\cdot\cos kt,\ e_n^-=\sin jx\cdot\sin kt\}\quad (j^2-k^2=-\mu_n^{-1}).$$

Define

$$E_n=\text{span}\{e_1^+, e_1^-, e_2^+, e_2^-, \cdots, e_n^+, e_n^-\}.$$

Clearly there exists a sequence of numbers: $0<R_1<R_2<\cdots$ such that

$$J(\varepsilon; u)\leqq 0 \quad \text{for all } u\in E_n \quad \text{with} \quad \|u\|_r\geqq R_n \quad \text{and for all } \varepsilon\in[0,1].$$

Let

$$B_R=\{u\in E; \|u\|_r\leqq R\}, \qquad D_n=B_{R_n}\cap E_n,$$
$$\Gamma_n=\{\gamma\in C(D_n, E); \gamma(T_\theta u)=T_\theta\gamma(u) \text{ for all } u \text{ and } \theta,\ \gamma(u)=u \text{ if } \|u\|_r=R_n\},$$
$$U_n=\{u=\tau e_{n+1}^+ + w; \tau\geqq 0,\ w\in B_{R_{n+1}}\cap E_n, \text{ and } \|u\|_r\leqq R_{n+1}\},$$
$$\Lambda_n=\{\lambda\in C(U_n, E); \lambda|_{D_n}\in\Gamma_n,\ \lambda(u)=u \text{ if } \|u\|_r=R_{n+1} \text{ or } u\in(B_{R_{n+1}}\backslash B_{R_n})\cap E_n\}.$$

Define for $n\in N$ and $\varepsilon\in[0,1]$,

$$b_n(\varepsilon)=\inf_{\gamma\in\Gamma_n}\sup_{u\in D_n} J(\varepsilon; \gamma(u)),$$
$$c_n(\varepsilon)=\inf_{\lambda\in\Lambda_n}\sup_{u\in U_n} J(\varepsilon; \lambda(u)).$$

The above definitions are analogous to those of P. H. Rabinowitz [3], which are used to prove the existence of solutions of the second order Hamiltonian systems.

It is clear that $c_n(\varepsilon)\geqq b_n(\varepsilon)$. In case $c_n(\varepsilon)>b_n(\varepsilon)$, as in [3], we have the following

Proposition 1. *For $\varepsilon\in(0,1]$, suppose that $c_n(\varepsilon)>b_n(\varepsilon)\geqq M$. Let $d\in(0, c_n(\varepsilon)-b_n(\varepsilon))$ and*

$$\Lambda_n(\varepsilon; d)=\{\lambda\in\Lambda_n; J(\varepsilon; \lambda(u))\leqq b_n(\varepsilon)+d \text{ on } D_n\}.$$

Define

$$c_n(\varepsilon; d)=\inf_{\lambda\in\Lambda_n(\varepsilon;d)}\sup_{u\in U_n} J(\varepsilon; \lambda(u)).$$

Then, $c_n(\varepsilon; d)$ is a critical value of $I(\varepsilon; u)$.

On the other hand, as in H. Brézis, J. M. Coron and L. Nirenberg [2], we have

Proposition 2. *For any $L>0$, there exists a constant $C_L>0$ independent of $\varepsilon\in(0,1]$ such that the assumption*

$$I'(\varepsilon; u)=0 \quad \text{and} \quad I(\varepsilon; u)\leqq L$$

imply

$$\|u\|_\infty\leqq C_L.$$

Recalling $\omega(\xi)=0$ for $|\xi|\leqq c_q$, for the proof of our theorem it suffices to prove the following

Proposition 3. *There exists a sequence $\{n_j\}_{j=1}^\infty$ such that for some constants $\delta_j\in(0,1]$ and $d_j>0$,*

$$c_{n_j}(\varepsilon)-2d_j\geqq b_{n_j}(\varepsilon)\geqq M \qquad \text{for all } \varepsilon\in(0,\delta_j].$$

Moreover, there exist sequence $\{m_j\}_{j=1}^\infty$ and $\{M_j\}_{j=1}^\infty$ which are independent of ε and

$$m_j \to \infty \qquad as\ j \to \infty,$$
$$m_j \leqq c_{n_j}(\varepsilon; d_j) \leqq M_j \qquad for\ \varepsilon \in (0, \delta_j].$$

This proposition follows from the next lemmas.

Lemma 2. *There is a constant $\beta > 0$ such that for $u \in E$ and $\theta \in [0, 2\pi)$,*
$$|J(0; T_\theta u) - J(0; u)| \leqq \beta(|J(0; u)|^{(q-1)/q} + 1).$$

Lemma 3. *For any $\delta > 0$ there is a constant $C_\delta > 0$ such that*
$$b_n(0) \geqq C_\delta n^{2(r-1)/(2-r)-\delta} \qquad for\ all\ n \in N.$$

Lemma 4. *There exists a sequence $\{n_j\}_{j=1}^\infty$ such that*
$$c_{n_j}(0) > b_{n_j}(0) \geqq M \qquad for\ all\ j \in N.$$

Lemma 5. *The functions $b_n(\cdot)$, $c_n(\cdot) : [0, 1] \to \boldsymbol{R}$ are right-continuous. In particular, they are continuous at* 0.

Here, as in K. Tanaka [5], we derive Lemmas 2, 3, 4, from (g_2), (g_3), (g_4) respectively. Lemma 5 is obtained from the fact that $J(\varepsilon; u)$ is a nondecreasing function of ε for fixed u.

Remark. It is clear that Theorem can be extended to the equation of the form:

$$v_{tt} - v_{xx} + g(x, v) = f(x, t).$$

In case that $g(x, t, v)$ depends also on t, we must act on $\boldsymbol{Z}_2$-symmetry as in K. Tanaka [5]. That is, we assume that $g(x, t, v)$ is odd in v and satisfies similar conditions to (g_1)–(g_4), then we have the existence result.

Acknowledgment. The author would like to thank Professor Haruo Sunouchi for his advice and encouragement.

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