4. Fock Space Representations of Virasoro Algebra and Intertwining Operators

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§0. In this note, we construct the Fock space representations of the Virasoro algebra $\mathcal{L}$ and intertwining operators between them in the explicit form, and give the analogous determinant formula for them as for the Verma modules (see V. G. Kac [4]). Proofs and details will be given in the forthcoming paper [6].

§1. The Virasoro algebra $\mathcal{L}$ is the Lie algebra over the complex number field $\boldsymbol{C}$ of the following form:

$$\mathcal{L}=\sum_{n\in \boldsymbol{Z}} \boldsymbol{C}e_n \oplus \boldsymbol{C}e_0',$$

with the relations: for any $m, n \in \boldsymbol{Z}$

$$\begin{cases}[e_n, e_m]=(m-n)e_{m+n}+((m^3-m)/12)\delta_{m+n,0}e_0', \\ [e_0', e_m]=0.\end{cases}$$

This is a unique central extension of the Lie algebra $\mathcal{L}'$ of trigonometric polynomial vector fields on the circle:

$$\mathcal{L}'=\sum_{n\in \boldsymbol{Z}} \boldsymbol{C}l_n\,;\ [l_n, l_m]=(m-n)l_{m+n}\ (m, n\in \boldsymbol{Z}) \qquad (l_n=z^{n+1}(d/dz)).$$

Let $\mathfrak{h}=\boldsymbol{C}e_0\oplus \boldsymbol{C}e_0'$ be the abelian subalgebra of $\mathcal{L}$ of maximal dimension. For each $(h, c)\in \boldsymbol{C}^2\cong \mathfrak{h}^*$ the dual of $\mathfrak{h}$, we can define the Verma module $M(h, c)$ and its dual $M^\dagger(h, c)$ as follows. $M(h, c)$ and $M^\dagger(h, c)$ are the left and right $\mathcal{L}$-modules with cyclic vectors $|h, c\rangle$ and $\langle c, h|$ with following fundamental relations respectively:

$$e_{-n}|h, c\rangle=0\ (n\geq 1)\,;\quad e_0|h, c\rangle=h|h, c\rangle,\quad e_0'|h, c\rangle=c|h, c\rangle\,;$$
$$\langle c, h|e_n=0\quad (n\geq 1)\,;\quad \langle c, h|e_0=\langle c, h|h,\quad \langle c, h|e_0'=\langle c, h|c.$$

V. G. Kac [4] studied these $\mathcal{L}$-modules and obtained the formula concerning the determinants of the matrices of their vacuum expectation values. By this Kac's determinant formula, F. L. Feigin and D. B. Fuks [3] determined the composition series of $M(h, c)$.

§2. Consider the associative algebra $\mathcal{A}$ over $\boldsymbol{C}$ generated by $\{p_n (n\in \boldsymbol{Z}), \Lambda\}$ with the following Bose commutation relations:

$$[p_m, p_n]=n\delta_{m+n,0}id\,;\quad [\Lambda, p_m]=0\quad (m, n\in \boldsymbol{Z}).$$

And consider the following operators in a completion $\hat{\mathcal{A}}$ of $\mathcal{A}$:

$$L_n=(p_0-n\Lambda)p_n+\frac{1}{2}\sum_{j=1}^{n-1} p_j p_{n-j}+\sum_{j\geq 1} p_{n+j}p_{-j} \qquad (n\geq 1)\,;$$

$$L_{-n}=(p_0+n\Lambda)p_{-n}+\frac{1}{2}\sum_{j=1}^{n-1} p_{-j} p_{j-n}+\sum_{j\geq 1} p_j p_{-n-j} \qquad (n\geq 1)\,;$$

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$$L_0=\frac{1}{2}(p_0^2-\Lambda^2)+\sum_{j\geq 1} p_j p_{-j};$$

$$L_0'=(-12\Lambda^2+1)id.$$

Then by long but elementary calculations, we get

Theorem 1. *The operators L_n $(n \in \boldsymbol{Z})$ and L_0' satisfy the commutation relations of the Virasoro algebra,* i.e. *for $m, n \in \boldsymbol{Z}$*

$$\begin{cases}[L_n, L_m]=(m-n)L_{m+n}+((m^3-m)/12)\delta_{m+n,0}L_0', \\ [L_0', L_m]=0.\end{cases}$$

§ 3. For each $(w, \lambda) \in \boldsymbol{C}^2$, we consider the left and right $\mathcal{A}$-module $\mathcal{F}(w, \lambda)$ and $\mathcal{F}^\dagger(w, \lambda)$ with cyclic vectors $|w, \lambda\rangle$ and $\langle\lambda, w|$ with following fundamental relations respectively:

$$p_{-n}|w, \lambda\rangle=0 \ (n\geq 1); \quad p_0|w, \lambda\rangle=w|w, \lambda\rangle, \quad \Lambda|w, \lambda\rangle=\lambda|w, \lambda\rangle;$$

$$\langle\lambda, w|p_n=0 \quad (n\geq 1); \quad \langle\lambda, w|p_0=\langle\lambda, w|w, \quad \langle\lambda, w|\Lambda=\langle\lambda, w|\lambda.$$

Then by using the canonical homomorphism π (i.e. $\pi(e_n)=L_n$ $(n \in \boldsymbol{Z})$; $\pi(e_0')=L_0'$), we get the left $\mathcal{L}$-module $(\mathcal{F}(w, \lambda), \pi_{(w,\lambda)}, \mathcal{L})$ which is called the Fock space representation, and by the explicit formulae of L_n and L_0', we get

$$\begin{cases}L_0|w, \lambda\rangle=(1/2)(w^2-\lambda^2)|w, \lambda\rangle; \ L_0'|w, \lambda\rangle=(1-12\lambda^2)|w, \lambda\rangle; \\ L_{-n}|w, \lambda\rangle=0 \qquad (n\geq 1).\end{cases}$$

By the universal property of the Verma module $M(h, c)$ as an $\mathcal{L}$-module, for each $(w, \lambda) \in \boldsymbol{C}^2$ we get the unique $\mathcal{L}$-module mapping

$$\pi_{w,\lambda}: M(h(w, \lambda), c(\lambda)) \longrightarrow \mathcal{F}(w, \lambda)$$

which sends the vacuum vector $|h(w, \lambda), c(\lambda)\rangle \in M(h(w, \lambda), c(\lambda))$ to the vacuum vector $|w, \lambda\rangle \in \mathcal{F}(w, \lambda)$, where

$$h(w, \lambda)=(1/2)(w^2-\lambda^2) \quad \text{and} \quad c(\lambda)=1-12\lambda^2.$$

Moreover this mapping $\pi_{w,\lambda}$ is degree-preserving, if we set

$$\deg e_n=\deg p_n=n \ (n \in \boldsymbol{Z}); \quad \deg e_0'=\deg \Lambda=0; \quad \deg|w, \lambda\rangle=\deg|h, c\rangle=0.$$

Denote by $C_d(w, \lambda)$ the determinant of the mapping $\pi_{w,\lambda}$ restricted to the degree $d(\geq 0)$ subspace $M_d(h, c)$ of dimension $p(d)$, where $p(d)$ is the number of partitions of the integer d.

Then by constructing intertwining operators (Theorem 4) and by showing their nontriviality (Theorem 5), we get the following.

Theorem 2. *For each $(w, \lambda) \in \boldsymbol{C}^2$, let $s_\pm$ be the roots of the equation $\lambda=(1/s)-(s/2)$, then*

$$C_d(w, \lambda)=\text{const.} \prod_{k=1}^{d} \prod_{a|k} \left(w+\frac{a}{2}s_+ + \frac{k}{2a}s_-\right)^{p(d-k)}.$$

As a corollary,

Theorem 3. (1) *The canonical $\mathcal{L}$-module mapping*

$$\pi_{w,\lambda}: M(h(w, \lambda), c(\lambda)) \longrightarrow \mathcal{F}(w, \lambda)$$

is isomorphic, if and only if the equation

$$(*) \qquad w+(a/2)s_+ + (b/2)s_- = 0$$

has no integral solutions $(a, b) \in \boldsymbol{Z}^2$ with $a\geq 1$ and $b\geq 1$.

(2) *The $\mathcal{L}$-module mapping $\pi_{w,\lambda}^\dagger: M^\dagger(h(w, \lambda), c(\lambda)) \rightarrow \mathcal{F}^+(w, \lambda)$ is isomorphic, if and only if the equation $(*)$ has no integral solutions $(a, b) \in \boldsymbol{Z}^2$*

with $a \leq -1$ *and* $b \leq -1$.

(3) $\mathcal{F}(w, \lambda)$ *is irreducible as an* $\mathcal{L}$*-module, if and only if the equation* $(*)$ *has no integral solutions* $(a, b) \in \boldsymbol{Z}^2$ *with* $ab \geq 1$.

And this condition (3) *is equivalent to the fact that the corresponding Verma module* $M(h(w, \lambda), c(\lambda))$ *is irreducible.*

§4. To construct intertwining operators between Fock spaces, we introduce the operators of the following type acting on $\mathcal{F}(w, \lambda)$. Fix $s \in \boldsymbol{C}^*$, and consider

$$X(s\,;\zeta)=\exp\left(s\sum_{n=1}^{\infty}\zeta^{n}\frac{p_n}{n}\right)\exp\left(-s\sum_{n=1}^{\infty}\zeta^{-n}\frac{p_{-n}}{n}\right)\zeta^{sp_0-(s^2/2)}T_s,$$

and for any $a \geq 1$

$$Z(s\,;\zeta_1,\cdots,\zeta_a)=F\left(\frac{s^2}{2}\,;\zeta_1,\cdots,\zeta_a\right)\exp\left(s\sum_{n=1}^{\infty}(\zeta_1^n+\cdots+\zeta_a^n)\frac{p_n}{n}\right)$$
$$\times\exp\left(-s\sum_{n=1}^{\infty}(\zeta_1^{-n}+\cdots+\zeta_a^{-n})\frac{p_{-n}}{n}\right)T_{as},$$

where

$$T_s : \mathcal{F}(w, \lambda) \longrightarrow \mathcal{F}(w+s, \lambda)$$

is the operator defined by

$$T_s|w, \lambda\rangle = |w+s, \lambda\rangle\,; \quad [T_s, p_n]=0 \ (n\neq 0)\,; \quad [T_s, \Lambda]=0,$$

and

$$F(\alpha\,;\zeta_1,\cdots,\zeta_a)=\prod_{j=1}^{a}\zeta_j^{-(a-1)\alpha}\prod_{1\leq i<j\leq a}(\zeta_i-\zeta_j)^{2\alpha}.$$

Operators of this type are called Vertex operators.

Then $X(s\,;\zeta)$ and $Z(s\,;\zeta_1,\cdots,\zeta_a)$ are multivalued holomorphic functions of $\zeta \in \boldsymbol{C}^*$ and $(\zeta_1,\cdots,\zeta_a) \in M_a$ respectively with valued in the operators acting on $\mathcal{F}(w, \lambda)$'s, where M_a is the manifold defined by

$$M_a=\{(\zeta_1,\cdots,\zeta_a)\in(\boldsymbol{C}^*)^a\,;\zeta_i\neq\zeta_j\ (1\leq i<j\leq a)\}.$$

And these operators satisfy the interesting formulae:

$$[L_m, X(s\,;\zeta)]=\zeta^{-m}\left(\zeta\frac{d}{d\zeta}-m\left(s\Lambda+\frac{s^2}{2}\right)\right)X(s\,;\zeta) \qquad (m\in\boldsymbol{Z},\ s,\zeta\in\boldsymbol{C}^*)\,;$$

$$[L_m, Z(s\,;\zeta_1,\cdots,\zeta_a)]$$
$$=\sum_{j=1}^{a}\zeta_j^{-m}\left[\zeta_j\frac{\partial}{\partial\zeta_j}+\left\{sp_0-\frac{a}{2}s^2-m\left(s\Lambda+\frac{s^2}{2}\right)\right\}\right]Z(s\,;\zeta_1,\cdots,\zeta_a)$$
$$(m\in\boldsymbol{Z},\ s\in\boldsymbol{C},\ (\zeta_1,\cdots,\zeta_a)\in M_a).$$

For each $\alpha \in \boldsymbol{C}^*$, denote by $\mathcal{S}_\alpha^*$ the local coefficient system with values in $\boldsymbol{C}$ which is determined by the monodromy of the multivalued holomorphic function $F(\alpha\,;\zeta_1,\cdots,\zeta_a)$ on M_a, and denote by $\mathcal{S}_\alpha$ the dual local system of $\mathcal{S}_\alpha^*$.

Fix $s \in \boldsymbol{C}^*$ and an integer $a \geq 1$, and take an element $\Gamma \in H_a(M_a\,;\mathcal{S}_\alpha)$. For each integer $b \in \boldsymbol{Z}$, we can consider the operator

$$\mathcal{O}(s, \Gamma\,; a, b)=\int_\Gamma Z(s\,;\zeta_1,\cdots,\zeta_a)\zeta_1^{-b-1}\cdots\zeta_a^{-b-1}d\zeta_1\cdots d\zeta_a.$$

Then we get the following.

Theorem 4. 1) *For each* $(w, \lambda) \in \boldsymbol{C}^2$, *the operator* $\mathcal{O}(s, \Gamma\,; a, b)$ *acts as*

$$\mathcal{O}(s, \Gamma\,; a, b) : \mathcal{F}(w, \lambda) \longrightarrow \mathcal{F}(w+as, \lambda).$$

2) *Take* $s \in \boldsymbol{C}^*$ *and* $a, b \in \boldsymbol{Z}$ *with* $a \geq 1$. *Put* $\lambda = \lambda(s) = (1/s) - (s/2)$, *then the operator*

$$\mathcal{O}(s, \Gamma; a, b) : \mathcal{F}(-(a/2)s - (b/s), \lambda) \longrightarrow \mathcal{F}((a/2)s - (b/s), \lambda)$$

commutes with the action of $\mathcal{L}$.

§ 5. The nontriviality of the obtained intertwining operators are guaranteed by the following theorem. Consider the set Ω_{a-1} defined by

$$\Omega_{a-1} = \{\alpha \in \boldsymbol{C};\ d(d+1)\alpha \notin \boldsymbol{Z},\ d(a-d)\alpha \notin \boldsymbol{Z}\ (1 \leq d \leq a-1)\}.$$

Then we get

Theorem 5. *There exists* $\Gamma(\alpha) \in H_a(M_a; \mathcal{S}_\alpha)$ *which depends holomorphically on* $\alpha \in \Omega_{a-1}$ *such that the operator*

$$\mathcal{O}(s; a, b) = \mathcal{O}(a, \Gamma(s^2/2); a, b) : \mathcal{F}(w - as, (1/s) - (s/2)) \longrightarrow \mathcal{F}(w, (1/s) - (s/2))$$

is nontrivial in the sense that for any $w \in \boldsymbol{C}$.

1) *For* $b \geq 0$, *the image* $\mathcal{O}(s; a, b)|w - as, (1/s) - (s/2)\rangle$ *is a nonzero vector.*

2) *For* $b < 0$, *there is a vector* $|v\rangle \in \mathcal{F}(w - as, (1/s) - (s/2))$ *such that* $\mathcal{O}(s; a, b)|v\rangle = |w, (1/s) - (s/2)\rangle$.

In order to prove this theorem, we explicitly construct a cycle $\Gamma(\alpha) \in H_a(M_a; \mathcal{S}_\alpha)$ parametrized by $\alpha \in \Omega_{a-1}$, by using the technique of resolutions of singularities.

The proof of the nontriviality of the operator $\mathcal{O}(s; a, b)$ can be reduced to the celebrated Selberg integral formula:

Theorem 6 (A. Selberg [5]). *Let* $\alpha, \beta, \gamma \in \boldsymbol{C}$ *satisfy the inequalities*

$$\operatorname{Re}\beta > -1, \quad \operatorname{Re}\gamma > -1, \quad \operatorname{Re}\alpha > -\min\left\{\frac{1}{m}, \frac{\operatorname{Re}\beta + 1}{m-1}, \frac{\operatorname{Re}\gamma + 1}{m-1}\right\},$$

then the improper integral (**) *converges absolutely and is explicitly expressed as*

$$(**) \qquad \int_{\Delta(m)} \prod_{1 \leq i < j \leq m} (k_i - k_j)^{2\alpha} \prod_{j=1}^{m} k_j^\beta (1 - k_j)^\gamma dk_1 \cdots dk_m$$
$$= \frac{1}{m!} \prod_{j=1}^{m} \frac{\Gamma(j\alpha + 1)\Gamma((j-1)\alpha + \beta + 1)\Gamma((j-1)\alpha + \gamma + 1)}{\Gamma(\alpha + 1)\Gamma((m + j - 2)\alpha + \beta + \gamma + 2)},$$

where $m = a - 1$ *and* $\Delta(m)$ *is the open* m-*simplex defined by*

$$\Delta(m) = \{(k_1, \cdots, k_m) \in \boldsymbol{R}^m;\ 1 > k_1 > \cdots > k_m > 0\}.$$

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