63. *A Characterization of Heegaard Diagrams for the 3-Sphere**)

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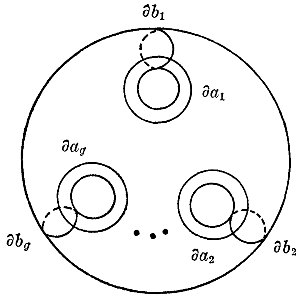
(Communicated by Kunihiko KODAIRA, M. J. A., June 10, 1986)

1. Introduction. In [7], Waldhausen showed the uniqueness of Heegaard splittings of genus g for the 3-sphere S^3. However, there are infinitely many Heegaard diagrams of each genus g (>1) for S^3 which look like quite different. No algorithm to distinguish those for S^3 among those for other 3-manifolds has been found in case of $g>2$ until now. There seems to be no good characterization of those for S^3 even if we accept non-algorithmic one. One of the essential difficulties comes from the fact that choice of meridian disks is free in both sides of the splittings. In this paper, we show a characterization (Theorem 2) which is not algorithmic but reduces the problem to the case of free choice of meridian disks in only one side.

To investigate Heegaard diagrams, their associated presentations of the fundamental groups have good information. Another result is a reduction (Theorem 3) of the cyclically reduced presentations associated with Heegaard diagrams for S^3 to those of the special type (S^3; ∂T_g, A, B, $\boldsymbol{a}$, $\boldsymbol{b}'$) (see the next section) where free choice of meridian disks is only in the side B as $\boldsymbol{b}'$. As an application of these results, we shall show a practical algorithmic characterization in the case of genus 2 in [5].

2. Statement of results. We follow [1] or [2] for the precise definitions of Heegaard splitting, diagram and sewing for a 3-manifold. Let T_g be a solid torus of genus g i.e., a 3-ball with g 1-handles. Assume that $S^3=R^3\cup\{\infty\}\supset R^3$ and T_g is embedded in R^3 as shown in Fig. 1.



$R^3\supset B=T_g$

Fig. 1

*) A part of this paper was done while the author was visiting at University of Toronto.

Let B be the image of the embedding of T_g and A be $S^3-\mathrm{Int}\,B$ (which is homeomorphic to T_g). Hereafter we identify T_g with B. Let $\boldsymbol{a}=\{a_1, \cdots, a_g\}$ and $\boldsymbol{b}=\{b_1, \cdots, b_g\}$ be the complete systems of meridian disks of A and B, respectively, whose boundaries ∂a_i and ∂b_i ($i=1, \cdots, g$) are as shown in Fig. 1. Let ϕ_0 be the orientation preserving homeomorphism on ∂T_g such that $\phi_0(\partial b_i)=\partial a_i$ ($i=1, \cdots, g$) (cf. $\phi_0=\mu_1\cdots\mu_g$ where μ_i is defined in [5]). We call ϕ_0 the standard Heegaard sewing of genus g for S^3.

Theorem 1. *Let ϕ be any Heegaard sewing of genus g for S^3. Then there exist homeomorphisms f and g on a solid torus T_g of genus g such that* 1) $\phi\sim f\phi_0 g$ (*$\sim$ denotes an isotopy*) *and* 2) *the induced isomorphism $f_\#$ on the fundamental group $\Pi(T_g)$ of T_g by f is the identity.*

Let $D=(M; F, V, W, \boldsymbol{v}, \boldsymbol{w})$ be an oriented Heegaard diagram of genus g for a 3-manifold M where F is a Heegaard surface, V and W are solid tori of genus g, and $\boldsymbol{v}$ and $\boldsymbol{w}$ are complete systems $\{v_1, \cdots, v_g\}$ and $\{w_1, \cdots, w_g\}$ of meridian disks of V and W, respectively. We denote the associated presentation of the fundamental group of M by $\Pi(D)$ $(=\langle x_1, \cdots, x_g; r_1, \cdots, r_g\rangle$ where each relator r_i is a cyclic word in the symbols $x_1, \cdots, x_g$ obtained by reading the intersection $\boldsymbol{v}\cap\partial w_i$ along ∂w_i) and the cyclically reduced one by $\tilde{\Pi}(D)$. Two cyclic words $\boldsymbol{w}, \boldsymbol{w}'$ are equivalent if one can be transformed to another by cyclic change of order and inversion. Then we denote this as $\boldsymbol{w}\equiv\boldsymbol{w}'$. Two presentations $P=\langle x_1, \cdots, x_m; r_1, \cdots, r_n\rangle$ and $P'=\langle x_1, \cdots, x_m; r_1', \cdots, r_n'\rangle$ are equivalent if $r_i\equiv r'_{j(i)}$ ($i=1, \cdots, n$) for some permutation $j(\ )$ of $\{1, \cdots, n\}$. We denote this also as $P\equiv P'$.

Theorem 2. *Let $D=(S^3; F, V, W, \boldsymbol{v}, \boldsymbol{w})$ be any Heegaard diagram of genus g for S^3. Then there exists a complete system $\boldsymbol{w}'$ of meridian disks of W such that the Heegaard diagram $D'=(S^3; F, V, W, \boldsymbol{v}, \boldsymbol{w}')$ for S^3 satisfies $\tilde{\Pi}(D')\equiv\langle x_1, \cdots, x_g; x_1, \cdots, x_g\rangle$.*

Remark. 1) This gives a characterization of Heegaard diagrams for S^3 (the sufficiency is due to [9, 8 or 4]).

2) For $g<3$, there is an algorithm to find such a Heegaard diagram D' from a given D (see [3], [5]).

Theorem 3. *Let $D=(S^3; F, V, W, \boldsymbol{v}, \boldsymbol{w})$ be any Heegaard diagram of genus g for S^3. Then there exists a Heegaard diagram $D'=(S^3; \partial T_g, A, B, \boldsymbol{a}, \boldsymbol{b}')$ of genus g for S^3 such that $\tilde{\Pi}(D)\equiv\tilde{\Pi}(D')$.*

3. Proof of Theorems. (1) *Proof of Theorem* 1. Let G_g be a set of Suzuki's homeomorphisms on T_g (i.e., $G_g=\{\rho, \rho_{12}, \omega_1, \tau_1, \theta_{12}, \xi_{12}\}$ for $g>2$, $G_2=\{\rho, \omega_1, \tau_1, \theta_{12}, \xi_{12}\}$ for $g=2$ and $G_1=\{\omega_1, \tau_1\}$ for $g=1$ where $\rho, \rho_{12}, \omega_1, \tau_1, \theta_{12}, \xi_{12}$ are defined in [6]) whose isotopy classes form a finite set of generators of the isotopy class group of all orientation preserving homeomorphisms on T_g (see [6]). Suppose that ϕ is orientation preserving. Then by [7], ϕ is isotopic to a composition $f_1\cdots f_k\phi_0 g_1\cdots g_n$ of homeomorphisms on T_g or ∂T_g where f_i (or f_i^{-1}), g_j (or g_j^{-1}) $\in G_g$ ($i=1, \cdots, k$, $j=1, \cdots, n$) and ϕ_0 is the standard Heegaard sewing of genus g for S^3 (see Proposition 2 of [2]). Let $f_{i1}, \cdots, f_{ik'}$ be the subsequence obtained from the sequence $f_1, \cdots, f_k$ by omitting f_i of the type τ_1, τ_1^{-1}, ξ_{12} or ξ_{12}^{-1}. Put $f'=f_1\cdots f_k$, $f_*=f_{i1}\cdots f_{ik'}$ and $g'=g_1\cdots g_n$. Then we have

$$\phi\sim f'\phi_0 g'=f'(f_*^{-1}f_*)\phi_0 g'=(f'f_*^{-1})f_*\phi_0 g'=(f'f_*^{-1})(\phi_0\phi_0^{-1})f_*\phi_0 g'$$
$$=(f'f_*^{-1})\phi_0(\phi_0^{-1}f_*\phi_0)g'.$$

Put $f=f'f_*^{-1}$ and $g=(\phi_0^{-1}f_*\phi_0)g'$. Since $\phi_0^{-1}f_{ij}\phi_0$ $(j=1,\cdots,k')$ can be extended to a homeomorphism on T_g by the definitions of ρ, ρ_{12}, ω_1, θ_{12} and ϕ_0 (see Section 3 in [6]), $g=(\phi_0^{-1}f_{i1}\phi_0)(\phi_0^{-1}f_{i2}\phi_0)\cdots(\phi_0^{-1}f_{ik'}\phi_0)g'$ also can be extended to a homeomorphism on T_g. By the fact of $\tau_{1\#}=\xi_{12\#}=1:\Pi(T_g)\to\Pi(T_g)$ (also see Section 3 in [6]), we have $f_\#=1$. Therefore we have the desired f and g. If ϕ is orientation reversing, we can apply the same argument to ϕr where $r: T_g\to T_g$ is an orientation reversing homeomorphism such that $r^2=1$. Then we have the desired ones by just replacing g with gr in the above argument.

(2) *Proof of Theorem* 2. Let $h_V: T_g\to V$ and $h_W: T_g\to W$ be homeomorphisms such that $h_V(\boldsymbol{b})=\boldsymbol{v}$ and $h_W(\boldsymbol{b})=\boldsymbol{w}$. Then $\phi=h_V^{-1}h_W|\partial T_g$ is a Heegaard sewing of genus g for S^3. By Theorem 1, we have $\phi\sim f\phi_0 g$. Put $h'_W=h_W g^{-1}: T_g\to W$ and $\boldsymbol{w}'=h'_W(\boldsymbol{b})$. Then we have a Heegaard diagram $D'=(S^3; F, V, W, \boldsymbol{v}, \boldsymbol{w}')$ whose corresponding Heegaard sewing ϕ' is $h_V^{-1}h'_W=h_V^{-1}h_W g^{-1}=\phi g^{-1}$. In [2], we introduced the notion of the presentation $\Pi(\psi)$ of $\Pi(M)$ associated with Heegaard sewing ψ of genus g for a 3-manifold M in order to study the presentation associated with Heegaard diagrams. By the fundamental results on $\Pi(\psi)$ in [2], we have $\tilde{\Pi}(D')\equiv\tilde{\Pi}(\phi')=\tilde{\Pi}(\phi g^{-1})\equiv\tilde{\Pi}((f\phi_0 g)g^{-1})=\tilde{\Pi}(f\phi_0)\equiv\tilde{\Pi}(\phi_0)\equiv\langle x_1,\cdots,x_g;x_1,\cdots,x_g\rangle$ because of $f_\#=1$.

(3) *Proof of Theorem* 3. Let h_V, h_W and ϕ be the same homeomorphisms defined in (2). We have $\tilde{\Pi}(D)\equiv\tilde{\Pi}(\phi)$ (see [2]). By Theorem 1, we have $\phi\sim f\phi_0 g$ and so $\tilde{\Pi}(D)\equiv\tilde{\Pi}(\phi)\equiv\tilde{\Pi}(f\phi_0 g)\equiv\tilde{\Pi}(\phi_0 g)$ because of $f_\#=1$. Let $h'_B: T_g\to B$ $(=T_g)$ be the homeomorphism g, $\boldsymbol{b}'$ be $h'_B(\boldsymbol{b})$ $(=g(\boldsymbol{b}))$ and D' be the Heegaard diagram $(S^3; \partial T_g, A, B, \boldsymbol{a}, \boldsymbol{b}')$. Let $h_A: T_g\to A$ be a homeomorphism such that $h_A|\partial T_g=\phi_0^{-1}$ and $h_A(\boldsymbol{b})=\boldsymbol{a}$. Then $h_A^{-1}h'_B|\partial T_g=\phi_0 g|\partial T_g$ is a Heegaard sewing corresponding to D' and we have $\tilde{\Pi}(D')\equiv\tilde{\Pi}(\phi_0 g)\equiv\tilde{\Pi}(D)$ similarly as before.

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