68. 2-nd Microlocalization and Conical Refraction

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(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1986)

§ 1. **Introduction.** The phenomenon of conical refraction has long been observed by physicists. It is attributed to the non-uniformity of multiplicities to Maxwell equation in the crystal and studied in the framework of Microlocal Analysis by Melrose-Uhlmann [8] and P. Laubin [5], [6].

We employ the theory of 2-microlocalization developed by M. Kashiwara and Y. Laurent (see [2], [4]) and gain a new insight about the phenomenon.

Explicitly, let P be a microdifferential operator defined in a neighborhood of $\rho_0 \in \sqrt{-1}\mathring{T}^*\boldsymbol{R}^n$, which satisfies the following conditions.

(1) P has a real principal symbol p.

Let $\Sigma_1=\{\rho \in \sqrt{-1}T^*\boldsymbol{R}^n\,;\, p(\rho)=0\}$ and $\Sigma_2=\{\rho \in \Sigma_1\,;\, dp(\rho)=0\}$.

(2) Σ_2 is a regular involutory submanifold in $\sqrt{-1}T^*\boldsymbol{R}^n$ through ρ_0 of codimension $d\geqq 3$.

(3) Hess $p(\rho)$ has rank d with positivity 1 if $\rho \in \Sigma_2$.

Moreover we assume

(4) P has regular singularities along Σ_2^C in the sense of Kashiwara-Oshima [3], where Σ_2^C denotes a complexification of Σ_2 in $T^*\boldsymbol{C}^n$.

Our main interest is the propagation of singularities on Σ_2 for the equation $Pu=0$, which can be transformed by a quantized contact transformation into

$$P_0u=\Big(D_1^2-\sum_{i,j=2}^{d} A^{ij}(x, D)D_iD_j+(\text{lower})\Big)u=0 \tag{5}$$

defined in a neighborhood of $\rho_1=(0, \sqrt{-1}dx_n)$. Here A^{ij} are of order 0 with $(\sigma(A^{ij}))$ positive definite. We remark that in this case $\Sigma_2=\{(x, \sqrt{-1}\xi)\,;\, \xi_1=\cdots=\xi_d=0\}$ and that P_0 has regular singularities along Σ_2^C.

We study (5) 2-microlocally along Σ_2. After transforming (5) by a quantized homogeneous bicanonical transformation, which is wider than quantized contact transformations, we give the canonical form of (5) as $D_1u=0$. Then we can easily obtain a theorem about the propagation of 2-microlocal singularities.

§ 2. **Notation.** Let X be a complex manifold and Λ be a regular involutory submanifold of T^*X. Λ is embedded naturally into $\Lambda\times\Lambda$. $\tilde{\Lambda}$ denotes the union of all bicharacteristics of $\Lambda\times\Lambda$ that pass through Λ. $\mathcal{E}_\Lambda^{2,\infty}$ is the sheaf on $T_\Lambda^*\tilde{\Lambda}$ of 2-microdifferential operators constructed by Y. Laurent [4].

Let M be a real analytic manifold whose complexification is X. Σ denotes a regular involutory submanifold of T_M^*X, whose complexification is Λ. $\tilde{\Sigma}$ denotes the union of bicharacteristics of Λ that pass through Σ. $T^*_\Sigma\tilde{\Sigma}$ is endowed with the sheaf C^2_Σ of 2-microfunctions constructed by M. Kashiwara. There exists the canonical spectral map

$$sp^2_\Sigma : \pi_\Sigma^{-1}(C|_\Sigma) \longrightarrow C^2_\Sigma \tag{6}$$

with

$$\pi_\Sigma : T^*_\Sigma\tilde{\Sigma}\backslash\Sigma \longrightarrow \Sigma.$$

For a microfunction u defined in a neighborhood of a point of Σ we define the 2-singular spectrum of u along Σ by

$$SS^2_\Sigma(u) = \mathrm{supp}\,(sp^2_\Sigma(u)). \tag{7}$$

See [2] for C^2_Σ.

§ 3. Statement of the result. We consider the equation $P_0 u = 0$. We put

$$\Sigma = \{(x, \sqrt{-1}\xi dx) \in \sqrt{-1}T^*\boldsymbol{R}^n\,;\ \xi_1 = \cdots = \xi_d = 0\} \tag{8}$$

and

$$\Lambda = \{(z, \zeta dz) \in T^*\boldsymbol{C}^n\,;\ \zeta_1 = \cdots = \zeta_d = 0\}. \tag{9}$$

We take a coordinate of $T^*_\Sigma\tilde{\Sigma}$ [resp. $T^*_\Lambda\tilde{\Lambda}$] as $(x, \sqrt{-1}\xi'', \sqrt{-1}x'^*)$ [(z, ζ'', z'^*)] with $\xi'' = (\xi_{d+1}, \cdots, \xi_n)$ and $x'^* = (x^*_1, \cdots, x^*_d)$ [resp. $\zeta'' = (\zeta_{d+1}, \cdots, \zeta_n)$ and $z'^* = (z^*_1, \cdots, z^*_d)$].

For a function g defined in an open set of $T^*_\Sigma\tilde{\Sigma}$, we define the relative Hamiltonian vector field of g by

$$H^r_g = \sum_{j=1}^{d} (\partial g/\partial x^*_j \cdot \partial/\partial x_j - \partial g/\partial x_j \cdot \partial/\partial x^*_j). \tag{10}$$

We announce

Theorem 1. *Let u be a microfunction solutions of* (5) *defined in a neighborhood of ρ_0. Then $SS^2_\Sigma(u)$ is invariant under H^r_f, where*

$$f = \sigma_\Lambda(P_0) \tag{11}$$

which is the principal symbol of P_0 along Λ. (*See* [4] *for definition.*)

We define the propagation cone of 2-microlocal singular support by

$$\tilde{\Gamma}_+ = \pi_\Sigma(\{\exp(sH^r_f)(0\,;\sqrt{-1}dx_n\,;x'^*)\,;\ f(0\,;\sqrt{-1}dx_n\,;x'^*) = 0,\ x^*_1 > 0,\ s \geqq 0\}). \tag{12}$$

Here $(0\,;\sqrt{-1}dx_n\,;x'^*)$ denotes a point of $\pi_\Sigma^{-1}((0\,;\sqrt{-1}dx_n))$ and $\exp(s\Theta)(\tau)$ is the exponential map for a vector field Θ starting from τ.

We give a microlocal Holmgren type theorem for (5).

Theorem 2. *There exists a neighborhood Ω of $\rho_0 = (0, \sqrt{-1}dx_n)$ such that for a microfunction solution u of* (5),

$$\Omega \cap \mathrm{supp}\,(u) \cap (\tilde{\Gamma}_+ \backslash \{\rho_0\}) = \emptyset \tag{13}$$

implies

$$SS(u) \not\ni \rho_0. \tag{14}$$

We remark that $\tilde{\Gamma}_+$ does not contain the inside of the cone and that Theorem 2 generalizes the result of Laubin [5], [6].

For details about this note, see Tose [14].

§ 4. Remark. In case $d=2$, the equation $Pu=0$ is studied by Tose [11], where (4) is not assumed. See also [12] and [13].

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