87. Pluricanonical Mappings of Elliptic Fiber Spaces

By Yoshio FUJIMOTO
Department of Mathematics, Kyoto University

(Communicated by Kunihiko KODAIRA, M. J. A., Oct. 13, 1986)

By an elliptic fiber space $f: V \to W$, we mean that f is a proper surjective morphism of a compact complex manifold V to a compact complex manifold W, where each fiber is connected and the general fibers are smooth elliptic curves. In particular, when V is a surface and W is a curve, we say that V is an elliptic surface over W.

By an n-dimensional elliptic fiber space $V \to W$ with $\kappa(V)=n-1$, we mean that the image of a rational map $\Phi_{|mK_V|}$ is $(n-1)$-dimensional for sufficiently large m. In this case if an m-th pluricanonical mapping $\Phi_{|mK_V|}: V \to \Phi_{|mK_V|}(V)$ is bimeromorphic to the original elliptic fiber space, we say that $\Phi_{|mK_V|}$ gives the Iitaka fibration.

Iitaka [4] showed that for any elliptic surface $f: S \to C$ with $\kappa(S)=1$, the m-th pluricanonical mapping $\Phi_{|mK_S|}$ gives the unique structure of the elliptic surface $f: S \to C$ if $m \geqq 86$. Moreover, he showed that 86 is the best possible number. On the other hand, Katsura and Ueno [5] showed that if S is an algebraic elliptic surface defined over an algebraically closed field k of characteristic $p \geqq 0$ with $\kappa(S)=1$, then $\Phi_{|mK_S|}$ gives the unique structure of the elliptic surface for every $m \geqq 14$.

One of the main purpose of the present note is to announce the existence of examples for elliptic fiber spaces. The details will be published elsewhere.

Our main results are the following:

Theorem 1. *Let $\{a_n\}_{n=1,2,\ldots}$ be a sequence of natural numbers defined as follows:*

$$a_1=2, \quad a_{n+1}=a_1 a_2 \cdots a_n+1,$$

and let $\{b_n\}_{n=1,2,\ldots}$ be a sequence of natural numbers defined as follows:

$$b_n=(n+1)(a_{n+3}-1)+2.$$

Then, for every positive integer n, there exists an elliptic fiber space $X^{(n+1)} \to \boldsymbol{P}^n$ over $\boldsymbol{P}^n$ which satisfies the following conditions.

(1) $\kappa(X^{(n+1)})=n$.

(2) *b_n is the best possible number of the Iitaka fibering of $X^{(n+1)}$, that is, $\dim|mK_X|=0$ if $m=b_n-1$, and the m-th pluricanonical mapping $\Phi_{|mK_X|}$ gives the Iitaka fibration for all $m \geqq b_n$.*

Moreover, $X^{(n+1)}$ is not in the class $\mathcal{C}$ in the sense of Fujiki [1]. *That is, $X^{(n+1)}$ cannot be bimeromorphic to any compact Kähler manifold.*

Examples. Now, we write down the first few terms of $\{a_n\}$ and $\{b_n\}$.

n	1	2	3	4	5	6
a_n	2	3	7	43	1807	3263443
b_n	86	5420	13053770	$\sim 10^{13}$	$\sim 10^{26}$	$\sim 10^{52}$

(1) $b_1=86$. This is the well known result of the elliptic surface. An elliptic surface $f: S \to \boldsymbol{P}^1$ over $\boldsymbol{P}^1$ with three multiple fibers of multiplicity $2, 3, 7$ and with constant moduli has the property that $\dim|85K_S|=0$.

(2) $b_2=5420$. This is the result obtained in [3]. There exists an elliptic threefold $f: X \to \boldsymbol{P}^2$ over $\boldsymbol{P}^2$ with constant moduli which has multiple fibers of multiplicity $2, 3, 7, 43$ along the four lines on $\boldsymbol{P}^2$ in a general position. X has the property that $\dim|5419K_X|=0$.

Though we have not proved the counterpart of Iitaka's theorem for elliptic fiber spaces, we conjecture that b_{n-1} is the best possible bound for all n-dimensional elliptic fiber spaces.

To prove Theorem 1, it is indispensable to study multiple fibers of elliptic fiber spaces and generalize the notion of a logarithmic transformation defined by Kodaira [6]. Our construction is as follows.

Let H_i $(1 \leq i \leq n+2)$ be $(n+2)$ hyperplanes on $\boldsymbol{P}^n$ which are in general position. Let $(a_1, a_2, \cdots, a_{n+2})$ be an $(n+2)$-tuple of positive integers defined as in Theorem 1. We perform logarithmic transformation along H_i's with multiplicity a_i. Then $X^{(n+1)} \to \boldsymbol{P}^n$ is an elliptic fiber space over $\boldsymbol{P}^n$ with constant moduli which has multiple fibers of multiplicity a_i along each H_i $(1 \leq i \leq n+2)$. Note that there exists *no* finite *abelian* covering of $\boldsymbol{P}^n$ which branches along H_i's $(1 \leq i \leq n+2)$ with the ramification index a_i respectively.

On the other hand, if we consider only *algebraic* elliptic fiber spaces, the best possible number of the Iitaka fibration seems to be much smaller than that of the analytic case. One of the main reasons is that the multiplicities of the multiple fibers of an algebraic elliptic fiber space with constant moduli should satisfy certain numerical conditions, as was shown by Katsura and Ueno [5]. Moreover, there is a deep connection with the theory of branched coverings of complex manifolds which was developed by Namba [7]. In [7], Namba obtained the necessary and sufficient condition for the existence of finite abelian coverings of $\boldsymbol{P}^n$. It is almost equivalent to the one obtained by Katsura and Ueno. Combining these two results, we see that an algebraic elliptic fiber space over $\boldsymbol{P}^n$ with constant moduli which has multiple fibers along hyperplanes can be constructed globally by taking finite abelian coverings of $\boldsymbol{P}^n$.

Our result is the following:

Theorem 2. *Let $\{c_n\}_{n=1,2,\ldots}$ be a sequence of natural numbers defined as follows:*

$$c_n=2(n^2+3n+3).$$

Then for every positive integer n, there exists an algebraic elliptic *fiber space $Y^{(n+1)} \to \boldsymbol{P}^n$ over $\boldsymbol{P}^n$ which satisfies the following conditions.*

(1) $\kappa(Y^{(n+1)})=n$.

(2) *c_n is the best possible number of the Iitaka fibration of $Y^{(n+1)}$, that is,* $\dim|mK_Y|=0$ *if $m=c_n-1$, and the m-th pluricanonical mapping $\Phi_{|mK_Y|}$ gives the Iitaka fibration for all $m \geqq c_n$.*

Examples. We write down the first few terms of $\{c_n\}$.

n	1	2	3	4	5	6
c_n	14	26	42	62	86	114

The construction is as follows. Let H_i's ($1 \leqq i \leqq n+2$) be $(n+2)$ hyperplanes on $\boldsymbol{P}^n$ which are in a general position. And let $(m_1, m_2, \cdots, m_{n+2}) = (2, \underbrace{2(n+2), \cdots, 2(n+2)}_{n+1})$ be a $(n+2)$-tuple of positive integers. Then performing logarithmic transformations along H_i's we construct an algebraic elliptic fiber space $Y^{(n+1)} \to \boldsymbol{P}^n$ with constant moduli which has multiple fibers of multiplicity m_i along each H_i. Note that, by Namba's theorem, there exists a finite abelian covering on $\boldsymbol{P}^n$ which branches along H_i's ($1 \leq i \leq n+2$) with the ramification index m_i respectively.

References

[1] A. Fujiki: On automorphism groups of compact Kähler manifolds. Invent. Math., **44**, 225–258 (1978).

[2] ——: On resolution of cyclic quotient singularities. Publ. of RIMS, **10**, 293–328 (1974).

[3] Y. Fujimoto: Logarithmic transformations on elliptic fiber spaces (preprint).

[4] S. Iitaka: Deformations of compact complex surfaces, ii. J. Math. Soc. Japan, **22**, 247–261 (1970).

[5] T. Katsura and K. Ueno: On elliptic surfaces in characteristic p. Math. Ann., **272**, 291–330 (1985).

[6] K. Kodaira: On compact analytic surfaces, ii–iii. Ann. of Math., **77**, 563–626, 1–40 (1963).

[7] M. Namba: Lectures on branched coverings and algebraic functions (preprint).